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THE MATHEMATICS STUDENT

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A NEW STUDY OF THE HALF-REGULAR CONTINUED FRACTION*

BY

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In this paper we propose to study the structure and properties of the several types of half-regular continued fractions by means of the transformations which convert a simple continued fraction into a half regular one [1] † of the type

$$l_o + \frac{\varepsilon_1}{b_1 +} \frac{\varepsilon_2}{b_2 +} \cdots\cdots \frac{\varepsilon_n}{+ b_n +} \cdots$$

where $|\varepsilon_n| = 1$, $b_n + \varepsilon_{n+1} \geqslant 1$, and b_n a positive integer > 1, if final.

For convenience, we adopt the following notations:

(i) $(b_0 ; b_1, b_2, \cdots) \equiv b_0 + \frac{1}{b_1 +} \frac{1}{b_2 +} \cdots$

(ii) $(b_n, \overline{b_{n+1}, b_{n+2}, \cdots} \overline{b_{n+r}}, b_{n+r+1})$
$$\equiv \frac{1}{b_n -} \frac{1}{b_{n+1} -} \frac{1}{b_{n+2} -} \cdots - \frac{1}{b_{n+r} +} \frac{1}{b_{n+r+1}}$$

(iii) $(b_{[r]}) \equiv \frac{1}{b+} \frac{1}{b+} \cdots \frac{1}{+b}$ (r terms)

(iv) $S_{r\ n} \equiv b_r + b_{r+1} + \cdots + b_{r+n-1}$, where $b_r, \cdots b_{r+n-1}$ are the successive partial quotients of the half-regular continued fraction $b_0 + \frac{\varepsilon_1}{b_1 +} \frac{\varepsilon_2}{b_2 +} \cdots\cdots$

§ 1. Sets of Transformations

1·1 E_1-transformation.

By repeated application of

(α) $\frac{1}{y} = 1 - \frac{1}{1+} \frac{1}{(y-1)}$ (traceable to Lagrange[2]) we can go on

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† This and other subsequent numbers in brackets appearing between consecutive lines relate to references at the end of the paper.

diminishing a partial quotient greater than 1 and increasing at the same time the *number* of partial quotients.

Thus, $(b_{r-1},\ b_r,\ \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot)$

$$= (b_{r-1}+1,\ \bar{1},\ b_r-1,\ \cdot\cdot\cdot\cdot\cdot\cdot\cdot)$$

$$= (b_{r-1}+1,\ \bar{2},\ \bar{1},\ b_r-2,\ \cdot\cdot\cdot\cdot\cdot)$$

$$= (b_{r-1}+1,\ \bar{2},\ 2,\ \bar{1},\ b_r-3,\ \cdot\cdot\cdot)$$

. . . .

$$= (b_{r-1}+1,\ \bar{2}_{[k]},\ \bar{1},\ b_r-k-1,\ \cdot\cdot\cdot),\ (k < b_r - 1) \quad \ldots \quad (1)$$

This transformation can evidently be continued up to $k = b_r - 2$, or $b - 3$ according as b_r is a medial partial quotient greater than 1 or a final quotient greater than 2, k being zero † or a positive integer. We call it the E_1-transformation. Here, the partial quotient b_r (>1) is replaced by $k+2$ new partial quotients, while b_r itself is reduced by $k+1$, the succeeding partial quotients, if any, being unaffected.

The maximum number of additional partial quotients that can thus be introduced is $b_r - 1$ or $b_r - 2$ according as b_r is medial or final; there are also just as many ways of applying the E_1-tranformation to the partial fraction $\frac{1}{+b_r}$.

We further notice that

$$S_{r-1,\ n} - n\ \ (= b_{r-1} + b_r - 2) \quad \ldots \quad (2)$$

is invariant under this transformation.

This property, being true for every E_1-transformation, is also true for any aggregate of E_1-transformations.

1·11 E_2, E_3, and P-transformations.

When the E_1-transformation is pushed to its extreme limit, we have

(i) $(b_{r-1},\ b_r) = (b_{r-1}+1,\ \bar{2}_{[b_r-3]},\ \bar{1},\ 2)$,

and (ii) $(b_{r-1},\ b_r,\ b_{r+1},\ \cdot\cdot) = (b_{r-1}+1,\ \bar{2}_{[b_r-2]},\ \bar{1},\ 1,\ b_{r+1},\ \cdot\cdot\cdot\cdot)$

called the extreme E_1-transformations.

† When $k=0$, the terms $\bar{2}_{[k]}$ simply drop out.

தமிழ்நாடு ஆவணக் காப்பக நூலகம்

We may now apply a variant of (α), viz.,

(β) $\frac{1}{1+}\ \frac{1}{y} = 1 - \frac{1}{y+1}$ to the last term in (i) and the terms

$\frac{1}{+1}\ \frac{1}{+b_{r+1}}$ in (ii), and obtain

$$(b_{r-1},\ b_r) = (b_{r-1}+1,\ \bar{2}_{[b_r-1]}) \qquad \ldots \quad (3)$$

called the E_2-transformation when b_r (>2) is a final quotient and

$$(b_{r-1},\ b_r,\ b_{r+1},\ \cdot\ \cdot\ \cdot\ \cdot\ ) = (b_{r-1}+1,\ \bar{2}_{[b_r-1]},\ \overline{b_{r+1}+1},\ \cdot\ \cdot\ \cdot\ \cdot\ ) \qquad \ldots \quad (4)$$

called the E_3-transformation when $b_r(>2)$ is a medial partial quotient.

The transformations (3) and (4) we call P-transformations when $b_r = 2$. ... (5)

It may be noted that the difference between the number of the partial quotients and their sum is increased by 1 on account of the E_2- or the final P-transformation and by 2 on account of the E_3- or the medial P-transformation.

1·12 New Convergents

From (3), we have

$(b_{r-1}+1,\ \bar{2}_{[k]}\ ) = (b_{r-1},\ k+1)$, so that

$(b_0\ ;\ b_1,\ b_2,\ \cdot\ \cdot\ \cdot\ b_{r-1}+1,\ \bar{2}_{[k]}\ )$ is the $(k+1)^{th}$ intermediate convergent between the principal convergents of the $(r-2)^{th}$ and r^{th} orders.

Again, $(b_{r-1},\ 1) = (b_{r-1}+1)$, and

$$(b_{r-1}+1,\ \bar{2}_{[k]},\ \bar{1}) = (b_{r-1}+1,\ \bar{2}_{[k-1]},\ \bar{1})$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$= (b_{r-1}),$$

and hence $(b_0\ ;\ b_1,\ b_2,\ \cdot\ \cdot\ \cdot\ b_{r-1}+1,\ \bar{2}_{[k]},\ \bar{1})$ is the principal convergent of the $(r-1)^{th}$ order, while $(b_0\ ;\ b_1,\ b_2,\ \cdot\ \cdot\ \cdot\ \cdot\ b_{r-1}+1)$ is the first intermediate convergent between the principal convergents of the $(r-2)^{th}$ and r^{th} orders.

We may now state the following results:

(i) The E_1-transformation applied to the rth partial fraction introduces after the principal convergent of the $(r-2)^{th}$ order the first $(k+1)$ intermediate convergents that come between the principal

convergents of the $(r-2)^{th}$ and r^{th} orders and then the principal convergents of the $(r-1)^{th}$ and r^{th} orders of the simple continued fraction.

(ii) The E_2-transformation which applies only to the last term of a finite simple continued fraction, just misses out the penultimate convergent and interpolates before the last principal convergent the entire gamut of intermediate convergents.

The E_3-transformation has a similar effect; but it preserves the next following principal convergent also.

(iii) The P-transformation applied to $\frac{1}{+2}$ (medial or final) simply substitutes an intermediate convergent for the preceding principal convergent and does not affect the other convergents. Thus, every one of the *new convergents* after the transformations is either a principal or intermediate convergent of the corresponding simple continued fraction.

N. B.—The E_1-, E_2-, and E_3-transformations are called extending transformations, since they increase the number of convergents in the new continued fraction; while the P-transformations preserve this number.

1·2 Annihilatory Transformations

There are two fundamental transformations that may be employed with reference to consecutive unit partial quotients:

(1) Those which affect the partial quotient immediately preceding the first unit partial quotient;

(2) those which do not do so.

We call them 'C_1', 'C_2' respectively, when successive unit partial quotients are either modified or annihilated as noticed below.

1·21 The C_1-transformation

We have

$$(b_r,\ 1_{[t]},\ b_{r+t+1},\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ )$$

$$=(b_r+1,\ \bar{2},\ 1_{[t-2]},\ b_{r+t+1}\ ,\ \cdot\ \cdot\ \cdot\ \cdot\ )\quad \text{by } (\beta)$$

$$=(b_r+1,\ \bar{3},\ \bar{2},\ 1_{[t-4]},\ b_{r+t+1}\ ,\ \cdot\ \cdot\ \cdot\ \cdot\ )$$

$$\cdot\ \cdot$$

$$\left.\begin{array}{l} = (b_r + 1,\ \bar{3}_{\left[\frac{t-2}{2}\right]},\ \bar{2},\ b_{r+t+1},\ \cdots\cdots)\ \text{when } t \text{ is even } \dagger \\ \text{or } (b_r + 1,\ \bar{3}_{\left[\frac{t-1}{2}\right]},\ \overline{b_{r+t+1} + 1}\ \cdots)\ \text{when } t \text{ is odd } \dagger\dagger \end{array}\right\} \ldots (6)$$

The expression $S_{r,n} + n$ is unaltered by this transformation; if $n = t + 2$ before the transformation, it is after the transformation equal to $\frac{t+4}{2}$ or $\frac{t+3}{2}$ according as t is even or odd. ... (7)

N. B.—This statement is true for the following transformation, only when t is éven.

1·22 The C_2-transformation

We write

$$(b_r,\ 1_{[t]},\ b_{r+t+1},\ \cdots\cdots)$$

$$= (b_r,\ 2,\ \bar{2},\ 1_{[t-3]},\ b_{r+t+1},\ \cdots\cdots)$$

$$= (b_r,\ 2,\ \bar{3},\ \bar{2},\ 1_{[t-5]},\ b_{r+t+1},\ \cdots\cdots)$$

. . . .

$$\left.\begin{array}{l} = (b_r,\ 2,\ \bar{3}_{\left[\frac{t-3}{2}\right]},\ \bar{2},\ b_{r+t+1},\ \cdots)\ \text{when } t \text{ is odd } \dagger\dagger\dagger (>1) \\ \text{or } (b_r,\ 2,\ \bar{3}_{\left[\frac{t-2}{2}\right]},\ \overline{b_{r+t+1} + 1},\ \cdots)\ \text{when } t \text{ is even } \dagger \end{array}\right\} (8)$$

1·23 The $c_1c_2^e$-transformation

A third method of annihilation is what may be called the eclectic method, which is a combination of the C_1-and C_2-methods applied partially to the unit partial quotients. It is important to note that the partial C_1-transformation applies only initially and absorbs an even number of consecutive unit partial quotients, while the partial C_2-transformation applies initially, medially and finally, absorbing an odd number of consecutive unit partial quotients in the first two cases and an odd or even number in the last case, symbolised accordingly by c_2° or c_2^{e}. The eclectic transformation can, therefore, be only one of four types;

(i) $c_1c_2c_2 \cdots\cdots c_2^{e}$; (ii) $c_1c_2c_2 \cdots\cdots c_2^{\circ}$;

(iii) $c_2c_2c_2 \cdots\cdots c_2^{e}$; (iv) $c_2c_2c_2 \cdots\cdots c_2^{\circ}$.

† When $t=2$ the term $\bar{3}$ drops out; †† similarly when $t=1$ and ††† when $t=3$.

Let t be the number of consecutive unit partial quotients affected by an eclectic transformation.

In the first two types, suppose c_1 absorbs k unit partial quotients and the c_2's, (say r in number) absorb respectively $l_1, l_2, \cdots\cdots l_r$ quotients. Then the reduction in the number of terms due to this transformation is given by

$$\frac{k}{2}+\sum_{i=1}^{r-1}\frac{l_i-1}{2}+\frac{l_r}{2}\text{ or }\frac{l_r-1}{2}=\frac{t-r+1}{2}\text{ or }\frac{t-r}{2} \qquad \dots (9)$$

according as l_r is even or odd, since $t=k+\sum_{i=1}^{r} l_i$

The maximum reduction will occur when $r=1$ or 2, viz., for the types $c_1c_2^e$, $c_1c_2^\circ$, $c_1c_2c_2^e$, and $c_1c_2c_2^\circ$.

Similarly, it can be shown that the maximum reduction for the types (iii) and (iv) will occur when $r=2$ or 3, viz., for

$c_2c_2^e$, $c_2c_2^\circ$, $c_2c_2c_2^e$, and $c_2c_2c_2^\circ$,

The scheme of reductions in the number of terms due to the application of the above eight special types of eclectic transformations is as follows:

Type	r	t	Reduction
$c_1c_2^e$	1	even	$t/2$
$c_1c_2c_2^\circ$	2	,,	$(t-2)/2$
$c_2c_2^\circ$	,,	,,	,,
$c_2c_2c_2^e$	3	,,	,,
$c_1c_2^\circ$	1	odd	$(t-1)/2$
$c_1c_2c_2^e$	2	,,	,,
$c_2c_2^e$	,,	,,	,,
$c_2c_2c_2^\circ$	3	,,	$(t-3)/2$

From the above tables we find that those which give the minimum number of partial quotients among the eclectic transformation are

(1) $c_1c_2^e$ when t is even,

and (2) $c_1c_2^\circ$, $c_1c_2c_2^e$, and $c_2c_2^e$ when t is odd, due to a reduction of $t/2$ terms in (1) and $\frac{t-1}{2}$ terms in (2).

But the C_1-transformation gives a reduction of $t/2$ terms when t is even and $\frac{t+1}{2}$ terms when t is odd, while the C_2-transformation gives the reduction of $t/2$ terms when t is even and $\frac{t-1}{2}$ terms when t is odd.

Hence, we may enunciate the result :

Of all the annihilatory transformations, the three transformations C_1, C_2, *and* $c_1c_2^e$ *alone give the least number of terms when* t *is even, while* C_1 *is the only such transformation when* t *is odd.* ... (10)

1·24 The Convergents of the annihilatory transformations

Applying (β) to the $(r+1)^{th}$ partial fraction of the simple continued fraction

(i) $(b_0 ; b_1, b_2, \cdots b_r, 1\ b_{r+2}, 1, b_{r+4}, \cdots\cdots)$

we get

(ii) $(b_0 ;\ b_1,\ b_2, \cdots b_r+1, \overline{b_{r+2}+1}, 1, b_{r+4}, \cdots)$

and note that the convergent of the rth order in (ii) is the same as that of the $(r+1)^{th}$ order in (i), while all the preceding convergents in (ii) are the same as the corresponding ones in (i).

Applying (β) again to the $(r+2)^{th}$ partial fraction in (ii), we have

(iii) $(b_0 ; b_1, b_2, \cdots b_r+1, \overline{b_{r+2}+2}, \overline{b_{r+4}+1}, \cdots\cdots)$

in which the convergent of the $(r+1)^{th}$ order is the same as that of the $(r+2)^{th}$ order in (ii) and that of $(r+3)^{th}$ order in (i).

Thus (β) applied to a unit partial fraction of (i) just annihilates only the preceding convergent, while it keeps intact all the others.

Now, the C_1 and C_2-transformations are merely aggregates of (β) transformations applied to alternate unit partial fractions of a set, commencing from the first or the second fraction of the set, and therefore annihilate the series of alternate convergents commencing from that which just precedes the one corresponding to the initial unit partial fraction annihilated.

Similarly, the eclectic transformations also annihilate sets of alternate convergents in a manner appropriate to their eclectic nature.

Thus, the annihilatory transformations have the property of contracting or reducing the number of terms, when they are applied. They do not introduce any intermediate convergents like E_1, E_2, E_3 and P-transformations, while they annihilate some of the principal convergents of the simple continued fraction.

1·25 *N. B.*—From § 1·12 and § 1·24, we notice the important fact that the last two convergents of a finite *h. r.* C. F are always equal to the last two convergents of the corresponding S.C.F except when the

transformation E_2 (or P) has been applied to the last partial fraction of the S.C.F (to convert it into the *h. r.* C.F. in question).

We call this the '*property of the last two convergents*' which holds even in the exceptional case, if we replace the last partial fraction $\frac{1}{-2}$ of the *h. r.* C.F by $\frac{1}{-1}\,\frac{1}{+1}$.

1·3 We have now fully envisaged the three groups (E_1, E_2, E_3), (P), and $(C_1, C_2, c_1c_2 \cdots c_2^e, c_1c_2 \cdots c_2^\circ, c_2, c_2 \cdots c_2^e, c_2c_2 \cdots c_2^\circ)$ of transformations which convert a simple continued fraction into a half-regular one. We will next prove the converse, viz., that *any* half-regular continued fraction can be derived from the corresponding simple continued fraction by applying one or more of the above transformations in a non-overlapping manner. Some of the important types that thus occur will be studied in § **3**.

1·4 Genesis of the half-regular continued fraction

Let $(b_0\,;\, b_1\,;\, b_2, \cdots b_r, \overline{b_{r+1}, \cdots \cdots b_s}, b_{s+1}, \cdots)$

be a *h. r.* C. F, which can be converted into the following C.F by Lagrange's Rule [2], i.e., a repeated application of (α):

$$(b_0\,;\, b_1, \cdots b_r - 1,\ 1,\ b_{r+1} - 2, \cdots 1,\ b_s - 1,\ b_{s+1}, \cdots)$$

where no element is negative but some may be zero†, as for example when $b_{r+1} = 2$. When one or more elements vanish, the C.F. admits of a condensation by the coalescence of the non-vanishing partial quotients before and after the zero; for obviously, $\frac{1}{a+}\,\frac{1}{0+}\,\frac{1}{b} = \frac{1}{a+b}$. Ultimately, after all the condensations are made, the result is a simple C.F. (S.C.F.)

If we now look at the process the other way, we easily find the transformations that have gone to alter the S.C.F. into the *h.r.* C.F. For example, when only two alternative elements have coalesced, either E_1 or P-transformation has been operative, while if successive alternate elements have coalesced E_1, E_2, or E_3 transformations must have taken effect. If there be no vanishing elements, evidently then one or more of the annihilatory transformations must have effected the change.

It will also be seen that these transformations have affected the partial quotients in order from left to right in such a non-

† No two zeroes will occur consecutively but they may occur in alternate places.

overlapping manner that when one of the transformations has altered a certain number of positive partial fractions (of the S. C. F.) into negative ones, the next partial fraction to be affected by a fresh transformation is that which comes after the last negative partial fraction.

We may call this '*the principle of non-overlapping*'. An example will make this clear. Consider the finite h. r. C. F.

$$1-\frac{1}{1}+\frac{1}{3}-\frac{1}{2}-\frac{1}{2}-\frac{1}{5}$$

which becomes after Lagrange's transformation

$$0+\frac{1}{1}+\frac{1}{0}+\frac{1}{2}+\frac{1}{1}+\frac{1}{0}+\frac{1}{1}+\frac{1}{0}+\frac{1}{1}+\frac{1}{4}$$

$$=\frac{1}{3}+\frac{1}{3}+\frac{1}{4} \quad \text{(S. C. F.)}$$

In getting back from the S. C. F. to the *h. r.* C. F, the transformations to be employed from left to right are in order: E_1, E_3; applied respectively to the first and second partial fractions (of the S. C. F).

The types of transformation in any particular case are also easily suggested by the characteristic sequence of partial quotients associated with them; thus, in the notation of page 1, a solitary $\bar{1}$ indicates E_1, a succession of $\bar{2}$'s followed by a partial quotient greater than 1 indicates E_3, a succession of pure $\bar{3}$'s implies C_1 and so on.

It is now plain how the several groups of transformations discussed here are sufficient to generate a half-regular continued fraction from the corresponding simple continued fraction.

§ 2. Properties of Residues

2·1. Lemma I: Of two consecutive residues in a *h. r.* C. F† in which the first is led by a unit partial fraction, either the absolute value of one is less and that of the other greater, than $\frac{1}{2}(\sqrt{5}-1)$ or both are absolutely equal to $\frac{1}{2}(\sqrt{5}-1)$. When two consecutive residues have their absolute values equal to $\frac{1}{2}(\sqrt{5}-1)$, the first of these is led by a unit partial fraction.

The proof is immediate, since

$$\frac{1}{1}+\frac{1}{x} \lesseqgtr \frac{\sqrt{5}-1}{2} \text{ according as } \frac{1}{x} \gtreqless \frac{\sqrt{5}-1}{2}$$

† This propositionis also true for the S. C. F. though certain qualifications are superfluous in this case

Again, if $|\varepsilon|=1$, $\frac{1}{a}+\frac{\varepsilon}{b}=\frac{\sqrt{5}-1}{2}$, and $\frac{1}{b}=\frac{\sqrt{5}-1}{2}$ we have $a=\frac{\sqrt{5}+1}{2}-\varepsilon.\ \frac{\sqrt{5}-1}{2}$, and equating the rational and irrational parts, we find $a=1$, $\varepsilon=1$.

Lemma II: If three consecutive partial fractions in a $h.\ r.$ C. F. occur as in $\frac{\pm 1}{2}\ \frac{1}{-\ 2\ \pm}\ \frac{1}{a\ \pm}\ \cdot\cdot\cdot\cdot$ then the absolute values of the residues led by them are not all less than $\frac{1}{2}\sqrt{(5-1)}$.

For, $\frac{1}{2}\ -\ \frac{1}{y} \lesseqgtr \frac{1}{2}(\sqrt{5}-1)$ according as $y \gtreqless 2+\frac{1}{2}(\sqrt{5}-1)$; and if $y=2+\frac{1}{a}+\cdot\cdot$, and $\frac{1}{2}-\frac{1}{y}<\frac{1}{2}(\sqrt{5}-1)$, then $\frac{1}{a}+\cdot\cdot\cdot>\frac{1}{2}(\sqrt{5}-1)$, while if $y=2-\frac{1}{a+}\cdot\cdot\cdot$, then $y<2+\frac{1}{2}(\sqrt{5}-1)$ and $\frac{1}{2-}\ \frac{1}{2-}\ \frac{1}{a}+\cdot\cdot$ is greater than $\frac{1}{2}(\sqrt{5}-1)$.

Cor: $\frac{1}{2}\ \frac{1}{-2}\ \frac{1}{-a\ \pm}\ \cdot\cdot\cdot$ and $\frac{1}{2}\ \frac{1}{-1}\ \frac{1}{+a\ \pm}\ \cdot\cdot\cdot$ are both greater than $\frac{1}{2}(\sqrt{5}-1)$ when $a\pm\cdot\cdot\cdot>1$.

2·2 The E_1-transformation introduces always one residue of the type $\frac{1}{1+}\ \frac{1}{a}$ or $\frac{1}{2}\ \frac{1}{-1}\ \frac{1}{+a}$, while E_2 and E_3-transformations always introduce one of the type $\frac{1}{2-}\ \frac{1}{2-}\ \frac{1}{a}$. Hence, a residue absolutely equal to or greater than $\frac{1}{2}(\sqrt{5}-1)$ occurs necessarily on account of any of the extending transformations.

2·3. The C_1-transformation, partial or otherwise, introduces residues of the types $\frac{1}{\pm 3\ -}$ or $\frac{1}{-\ 2\ +}$, which are both numerically less than $\frac{1}{2}$ and therefore less than $\frac{1}{2}(\sqrt{5}-1)$.

The C_2-transformation, partial or otherwise, introduces also residues less than $\frac{1}{2}$ except in the case of the first residue, which is led by $\frac{1}{2-}$. This first residue is equal to (1_{∞}) or $(1_{[r]},\ \cdot\cdot\cdot)$ where r is the number of unit partial quotients absorbed by this C_2-transformation and all the following ones (if any) in the particular eclectic transformation considered.

Now $(1_{[r]},\ a) \gtrless (1_{\infty})=\frac{1}{2}(\sqrt{5}-1)$ according as r is odd or even, when $a\geqslant 2$. Obviously, r is odd for at least one c_2-transformation

in any eclectic transformation which either involves more than one partial c_2-transformation or is of the type $c_1c_2^\circ$, the number of unit partial quotients absorbed by c_2° or $c_2c_2^e$ being always odd; and r is certainly even for the transformation $c_1c_2^e$ as well as the complete C_2-transformation absorbing an even number of unit partial fractions.

Hence, a C_2-transformation, partial or otherwise, introduces one or more residues equal to or greater than $\frac{1}{2}(\sqrt{5}-1)$ in all cases except where it occurs once and once only and absorbs a finite and even number of unit partial quotients. It is therefore remarkable that *those annihilatory transformations which give the least number of partial quotients are also just those which make the residues absolutely less than* $\frac{1}{2}(\sqrt{5}-1)$. ... (11)

2·4 As regards the medial P-transformation, it introduces (vide Lemma II) a residue, absolutely equal to or greater than $\frac{1}{2}(\sqrt{5}-1)$ only under the following circumstances:

(i) *When the partial* $\dfrac{1}{+2}$ *to which it is applied, immediately succeeds a unit partial fraction, and*

(ii) *When the residue in the S.C.F. succeeding* $\dfrac{1}{+2}$ *is either equal to* $\frac{1}{2}(\sqrt{5}-1)$ *or in the form* $(1_{[r]}, a, \cdots)$ *where* $a \geqslant 2$ *and* r *odd.* (12)

Hence, in order that a P-transformation may secure a residue absolutely less than $\frac{1}{2}(\sqrt{5}-1)$ in a *h. r.* C.F., the unit partial fractions (if any) just preceding $\dfrac{1}{+2}$ must first be dealt with by appropriate annihilatory transformations and then the P-transformation may be applied to $\dfrac{1}{+2}$ provided it survives the antecedent annihilatory transformations and is not followed immediately by an odd number or an infinite number of unit partial quotients. Indeed, the partial fraction $\dfrac{1}{+2}$ will not survive, if it is either immediately followed by an infinite or odd number of consecutive unit partial fractions or immediately preceded by unit partial fractions, when these are so annihilated that the residues are absolutely less than $\frac{1}{2}(\sqrt{5}-1)$, unless the immediately preceding unit partial fractions (if any) are even in number and are operated upon by the C_1-transformation. When the batch of unit partial fractions immediately following $\dfrac{1}{+2}$ is even, the application of the P-transformation first and then the C_1-transformation is

tantamount to applying first the C_2-transformation to the even number of unit partial fractions and then applying the P-transformation to $\frac{1}{+2}$ which will not have been affected by the C_2-transformation. Thus, there is nothing lost by applying the P-transformations after the annihilatory transformations C_1, C_2, $c_1c_2^e$ have taken effect appropriately, in the process of conversion of a S.C.F. into a h.r. C.F. with all residues absolutely less than $\frac{1}{2}(\sqrt{5}-1)$.

2·5 Summing up, we note that the extending transformations always introduce a residue numerically equal to or greater than $\frac{1}{2}(\sqrt{5}-1)$, while among the annihilatory transformations, only C_2, $c_1c_2^e$ applied to a batch of even number of consecutive unit partial fractions and C_1 applied to any batch (finite or infinite) of such fractions introduce residues all absolutely less than $\frac{1}{2}(\sqrt{5}-1)$; but the P-transformation will introduce residues absolutely equal to or greater than $\frac{1}{2}(\sqrt{5}-1)$, only when the partial fraction to which it is applied comes immediately after a unit partial fraction or before a batch of an odd number or infinite number of such fractions. In respect of the combination of the P and C_1, C_2, $c_1c_2^e$ transformations securing the absolute values of all residues less than $\frac{1}{2}(\sqrt{5}-1)$, it will be convenient to apply the appropriate annihilatory transformations first and then the P-transformations optionally to the surviving partial fractions of the type $\frac{1}{+2+}$ preceded and followed by partial quotients greater than 1, without violating 'the principle of non-overlapping' observed in § 1·4 *supra.*

§ 3. Varieties

3·1 Vahlen's Theorem(3): The reduced fraction p/q $(q>0)$ admits of exactly q different ways of development as a finite half-regular continued fraction.

Let p/q be expressed as the S.C.F.

$$F_n \equiv (b_0;\ b_1,\ b_2,\ \cdots\cdots\ b_n)$$

where $b_n \geqslant 1$; and let B_r be the number of ways in which $F_r(\equiv(b_0;\ b_1 \cdots\cdots b_r))$ can be developed into finite h. r. C.F.'s.

The required h. r. C.F. developments of F_n (including F_n) can be divided into two classes:

I. Those which involve one of our transformations (β, E_1, E_2, E_3, P, and the identical transformation) *affecting* the last partial fraction to which it is not directly applied; in other words, the

transformation is applied to the penultimate partial fraction so as to affect the last partial fraction.

II. Those which involve a transformation applied to the last partial fraction. The processes by which these two classes are respectively derived will be referred to as the *first* process and the *second* process.

If $b_n > 1$, we may apply to $\frac{1}{+b_{n-1}}$

(i) E_3-transformation when $b_{n-1} > 2$, or

(ii) P- ,, ,, $b_{n-1} = 2$, or

(iii) β- ,, ,, $b_{n-1} = 1$

these three being the only transformations which affect also b_n.

With every one of the B_{n-2} h.r. C.F's equal to F_{n-2}, modified if necessary, so as to possess '*the property of the last two convergents*' (vide § 1·25), we may associate one of the above transformations and derive the same number of h.r. C.F's equal to F_n.

This is the *first* process.

Next, if $b_n > 2$, we may apply to $\frac{1}{+b_n}$ the identical transformation (i.e. keep it intact) or the E_1-transformation in one of $b_n - 2$ ways, or the E_2-transformation and associate each of these b_n different ways on the whole with each of B_{n-1} *h.r.* C.F's equal to F_{n-1} *modified as before*, so as to obtain $b_n B_{n-1}$ different *h.r.* C.F's equal to F_n.

When $b_n = 2$, we may apply to $\frac{1}{+2}$ either the P-transformation or the identical transformation and so deal with it, again, in $b_n(=2)$ ways and by association obtain as before $b_n B_{n-1}$ different h.r. C.F's equal to F_n.

This is the *second* process.

On the whole, from the two processes together, we can derive $B_{n-2} + b_n B_{n-1}$ different h.r. C.F's equal to F_n.

When $b_n = 1$, we can either add 1 to the last partial quotient of each of the B_{n-1}, h.r. C.F's equal to F_{n-1}, modified as aforesaid where necessary or associate the partial fraction $\frac{1}{+(b_{n-1}+1)}$ with each of the B_{n-2} h.r. C.F's equal to F_{n-2}, modified as before. We get thus $B_{n-2} + b_n B_{n-1}$ different h.r. C.F's again as before.

The associations mentioned here are valid on account of '*the property of the last two convergents*'. For, if

$$(b_0\,;\,b_1,\,b_2,\,\cdot\,\cdot\,\cdot\,\cdot\,b_r) = c_0 + \frac{\varepsilon_1}{c_1} + \frac{\varepsilon_2}{c_2} + \cdot\,\cdot \quad + \frac{\varepsilon_k}{c_k} \text{ (where } |\varepsilon_{k'}| = 1,\ k' = 1, 2 \cdot\cdot k)$$

such that the last two convergents of the two continued fractions are equal each to each, then,

$$(b_0\,;\,b_1,\,b_2,\,\cdot\,\cdot\,\cdot\,b_r,\,b_{r+1},\,\cdot\,\cdot\,\cdot\,) = c_0 + \frac{\varepsilon_1}{c_1} + \frac{\varepsilon_2}{c_2} + \cdot\,\cdot\,\cdot + \frac{\varepsilon_k}{c_k} + \frac{1}{b_{r+1}} + \cdot\,\cdot\,\cdot\,\cdot\,\cdot$$

for the convergent of the $(r+t)^{th}$ order on the *l.h.s.* is equal to that of the $(k+t)^{th}$ order on the *r.h.s.*, for $t = 1, 2, \cdot\,\cdot\,\cdot\,\cdot\,\cdot$

We may now state the result:

If $F_r = (b_0\,;\,b_1,\,b_2,\,\cdot\,\cdot\,\cdot\,b_r)$ and B_r the number of *h.r.* C.F's equal to F_r, then $B_n = B_{n-2} + b_n\,B_{n-1}$. ... (13)

Obviously, $B_0 = 1$, and $B_1 = b_1$ (vide '*Second*' process).

Hence $B_n = K(1, n)^{(4)} = q$ (in the notation of continuants)

This proves Vahlen's theorem.

Note: Though this proof is longer than the usual one (vide Perron, p. 157), it has the advantage of giving a better insight into the structure of the h.r. C.F's and their enumeration.

3·2 We will next discuss the relative lengths of the h.r. C.F's equal to $F_n\left(= b_0 + \frac{1}{b_1} + \cdot\,\cdot\,\cdot + \frac{1}{b_n}\right)$, $(b_n > 1)$, the length being defined by the number of partial fractions in a continued fraction.

Let the partial fractions of F_n to which one or more of the transformations of § 1 may be applied to convert it into a *h.r.* C.F. be in order

$$\frac{1}{+b_{i1}},\ \frac{1}{+b_{i2}},\ \cdot\,\cdot\,\cdot\,\cdot\,\cdot\,\cdot,\ \frac{1}{+b_{ik}}$$

where $i_1 < i_2 < \cdot\,\cdot\,\cdot\,\cdot\,\cdot\,\cdot < i_k \leqslant n$

When $b_{i_r} = 1$ $(r \leqslant k)$, the β-transformation just annihilates it.

When $b_{i_r} > 1$, the transformations E_2, E_3, P (wherever they are applicable) replace the term by exactly $(b_{i_r} - 1)$ terms; while the E_1-transformation replaces it by not more than b_{i_r} or $b_{i_r} - 1$ terms according as it is applied medially or finally.

Hence, the total effect of the transformations is such that the number of partial fractions in a $h.\ r.$ C. F development can never exceed

$$n-k+\sum_{k'=1}^{k} b_{i_{k'}} \text{ or } n-k-1+\sum_{k=1}^{k} b_{i_{k'}}$$

according as $i_k < n$ or $i_k = n$

Since $b_n > 1$, $\sum_{1}^{n} b_r \geqslant n-k+b_{i_1}+b_{i_2}+\cdots+b_{i_k}$ if $i_k = n$; and

$$\sum_{1}^{n} b_r > n-k+b_{i_1}+b_{i_2}+\cdots+b_{i_k} \text{ if } i_k < n$$

$$\geqslant n-k+1+b_{i_1}+b_{i_2}+\cdots+b_{i_k}.$$

In all cases, therefore, the upper limit for the number of partial fractions in a finite h. r. C. F development of F_n is

$$b_1+b_2+\cdots+b_n-1=S_{1,n}-1 \qquad \dots (14)$$

3·21. From (14) we infer that the longest finite h. r. C. F development of F_n cannot have more than $S_{1,n}-1$ partial fractions. We will now prove that there *two* and only *two* longest h. r. C. F's having exactly $S_{1,n}-1$ partial fractions.

To attain the maximum number, the only applicable transformations are E_1 and E_2 (or P). For, the E_3-transformation, applied, say to $\frac{1}{+b_{i_{k'}}}$ $(k' \leqslant k)$ replaces it and the next partial fraction by just $b_{i_{k'}}$ $(< b_{i_{k'}} + b_{ik'+1})$ terms; while the P-transformation as well as the annihilatory transformation applied to the same replaces it by less than $b_{i_{k'}}$ terms, when $b_{i_{k'}}=2$ or 1. Further, the extreme E_1-transformation applied *medially* gives rise to just as many new partial fractions as the units in the partial quotient replaced, while it gives *finally* just one less than this number in the same way as the E_2 (or P) transformation. The two longest (in the sense, not shorter than any other) $h.\ r.$ C. F's are therefore obtained by applying the extreme E_1-transformation to all the partial fractions except the last (barring unit partial fractions) and transforming the last either by the extreme E_1-transformation or the E_2 (or P) transformation. In each of these two cases the number of partial fractions is the maximum possible, viz. $b_1+b_2+\cdots+b_n-1$ and the residues are all numerically greater than $\frac{1}{2}$ except the last which is numerically equal to $\frac{1}{2}$.

Further, the difference between the number of the partial quotients of either of the longest $h.\ r.$ C. F's and their sums is either

equal to or one more than the corresponding difference of the S. C. F_1, according as the E_2 (or P) transformation is not or is employed.

3·3. The reduced half-regular continued fraction (reduziert regelmässigen Kettenbrüchen) ... (1)

A reduced *h. r.* C. F is one in which all the partial fractions are negative[6]; this may be briefly referred to as r. r. C. F.

To convert $F_n=(b_0;\ b_1,\ b_2, \cdots b_n)$ a S. C. F into a r. r. C. F we employ the E_3 (or P or β) transformation with respect to the alternate terms

$$\frac{1}{+b_1}, \frac{1}{+b_3}, \quad \cdots \text{ and } \frac{1}{+b_{n-2}} \text{ or } \frac{1}{+b_{n-1}}$$

according as n is odd or even and further apply the E_2 (or P) transformation to the last partial fraction only when n is odd. For example, $(b_0;\ b_1,\ b_2, \cdots b_{2n})$

$$=(b_0+1;\ \bar{2}_{[b_1-1]},\ \overline{b_2+2}, \cdots \bar{2}_{[b_{2n-1}-1]},\ \overline{b_{2n}+1}). \qquad \dots (1)$$

$$(b_0;\ b_1,\ b_2, \cdots b_{2n+1})$$

$$=(b_0+1;\ \bar{2}_{[b_1-1]},\ \overline{b_2+2}, \cdots \bar{2}_{[b_{2n-1}-1]},\ \overline{b_{2n}+2},\ \bar{2}_{[b_{2n+1}-1]}). \quad \dots (2)$$

If the S. C. F. be infinite, the corresponding *r.r.* C.F is

$$(b_0+1;\ \bar{2}_{[b_1-1]},\ \overline{b_2+2},\ \bar{2}_{[b_3-1]},\ \overline{b_4+2}, \cdots \text{ ad inf.}) \qquad \dots (3)$$

With the help of examples (1) and (2), it is easy to verify the following:

If (S, N), (S', N') *represent within their brackets the sum and the number of partial quotients of a finite simple continued fraction and the corresponding reduced regular continued fraction then* (*i*) $S'-S=N'-1$, *whether* N *is odd or even; and* (*ii*) *the sum of the partial quotients of the S. C. F. in the even places is equal to* N' *or* $N'-1$ *according as* N *is even or odd.* ... (15)

N. B.—Even if the first partial quotient is 0, it must be counted as one of the partial quotients for the purpose of the above result.

3·4 The Shortest half-regular continued fractions.

The shortest *h.r.* C.F. development of a given rational number cannot contain partial fraction groups of the following types:

$$(1)\ \frac{1}{\pm 1};\quad (2)\ \frac{1}{\pm a}+\frac{1}{2}-\frac{1}{2}\pm\frac{1}{b};\quad (3)\ \frac{1}{\pm a}-\frac{1}{2}-\frac{1}{2}\pm\frac{1}{b}.$$

For, all these are capable of reduction to a lesser number of terms. In fact (1) can be annihilated by (β), (2) can be reduced to $\frac{1}{(a+1)} - \frac{1}{3 \mp} \frac{1}{(b \pm 1)}$, and (3) to $\frac{1}{(a-1)} + \frac{1}{3 \mp} \frac{1}{(b \pm 1)}$ (the upper signs going together and similarly the lower ones).

The transformations E_1, E_2, E_3 always introduce partial fractions of the above kind, especially types (1) and (3) and therefore cannot give rise to the shortest *h.r.* C.F development.

As regards unit partial fractions, we have seen that the maximum reduction in their number is achieved by the transformations C_1, C_2, $c_1c_2^e$ appropriately applied according to (10). With reference to the P-transformation which does not affect the number of terms, it may be noticed that if it is applied to $\frac{1}{+2}$ adjoining unit partial fractions (if any) before the transformation C_1, C_2, or $c_1c_2^e$ has taken effect according to (10), the result will be to reduce the number of adjoining unit partial fractions by 1 or 2 and thereby modify the maximum reduction possible in the number of these unit partial fractions according to the following scheme:

Number of unit partial fractions		**Maximum reductions possible as per (10).**	
Before	After	Before	After
the P-transformation		the P-transformation	
$2n$	$2n-1$	n	n
$2n$	$2n-2$	n	$n-1$
$2n+1$	$2n$	$n+1$	n
$2n+1$	$2n-1$	$n+1$	n

Thus the effect of the premature P-transformation is not to reduce the S. C. F. to the shortest *h.r.* C.F except when it applies just to only one partial fraction $\frac{1}{+2}$ adjoining an even batch of unit partial fractions. In the exceptional case, it is easily seen that the effect of applying the P-transformation first and then the C_1-transformation is the same as applying first a C_1 or C_2 transformation and then the P-transformation according as $\frac{1}{+2}$ adjoins the batch of unit partial fractions on the right or on the left. Hence, to obtain the shortest *h.r.* C.F development, we may well postpone the P-transformation which is optional until all the appropriate annihilatory transformations have taken place so as to effect maximum reduction in the number of terms. As this process is not generally unique,

there are often several half-regular C. F. developments of the *same* length but shorter than others, for a given finite S. C. F.

Provided the P-transformations be nowhere applied in the above process the sum of the partial quotients as well as their number is the same for all such shortest C. F. developments, while the sum of these two quantities is equal to the corresponding sum for the simple continued fraction. (*Vide* (7)). ... (16)

In this connection, we may note that the number of ways in which an even number (say $2n$) of successive unit partial quotients occurring between two partial quotients greater than 1 can be reduced to the minimum number n in $(n+1)$ ways, which comprise C_1-, C_2-, and $(n-1)$, $c_1c_2^e$ transformation

Example : $$2+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2}$$

can be transformed into one of the shortest *h.r.* C.F's without applying the P-transformations, 3×2 or 6 ways, for the two batches of four and two unit partial fractions can be modified respectively in 3 ways and 2 ways which may be associated one with the another, while the odd batch can be modified only in one way i.e. by applying the C_1-transformation.

The total number of ways of obtaining the shortest *h.r.* C.F's equal to a given S. C. F. is discussed by Tietze[5] and the discussion need not be reproduced here.

3·41 Among the equilong shortest *h.r.* C.F developments *two* are well known :

(1) the continued fraction to the nearest integer (nach nachsten Ganzen) in which the residues are never absolutely greater than $\frac{1}{2}$;

(2) the singular continued fraction (due to Hurwitz).

The first of these is obtained by applying the C_1-transformation uniformly to all the batches of unit partial fractions and the *h.r.* C.F so obtained, may be represented by

$$\left(c_0\,;\ \frac{\varepsilon_n}{c_n}\right)_{n=1}^{n=k},$$

which is characterised by the properties

(i) $c_n > 1$ $(n \leqslant k)$ and (ii) $c_n + \varepsilon_{n+1} > 1$ $(n < k)$,

easily verified by (6).

The singular continued fraction is derived by employing the C_1-transformation for odd batches and the C_2-transformation for even batches of unit partial fractions. It is characterised by the properties (i) $c_n > 1$ and $c_n + \varepsilon_n > 1$ $(n < k)$ verified by (6) and (8).

If the continued fractions be infinite, the restrictions for n have to be removed.

3·42 From § 2·5, it will be clear that the transformations used in converting a simple continued fraction into one of the shortest half-regular continued fractions are the same as those which secure the residues absolutely less than $\frac{1}{2}(\sqrt{5}-1)$. Hence, *all h.r. C.F developments of a rational number of the same length as the corresponding C. F. to the nearest integer, are characterised by the property that its absolute residues are less than* $\frac{1}{2}(\sqrt{5}-1)$. (Tietze)[5] ... (17)

Perron calls such continued fractions 'kanonisch'.

3·5 The number of ways of applying the annihilatory transformations to a batch of t consecutive unit partial quotients preceded and followed by partial quotients greater than 1, (occurring in a S. C. F.) will now be discussed.

The annihilatory transformations operate in one of *four* following ways:

(i) by affecting the t unit partial quotients only;

(ii) " $(t+1)$ consecutive partial quotients commencing from the one just preceding the first unit partial quotient;

(iii) " $(t+1)$ consecutive partial quotients ending with the one just following the last unit partial quotient;

and (iv) " $(t+2)$ consecutive partial quotients including the two partial quotients adjoining on either side the batch of t unit partial quotients.

The total number of ways of annihilation accordingly corresponds to the sum of the number of ordered partitions of

(1) t, (2) $t+1$ taken twice (3) $t+2$

into a certain number of 3's and 2's, the first part being always 3. This is easily seen from the fact that the effect of the transformations C_1 or C_2 (partial or otherwise) is to modify the first three consecutive terms and the succeeding terms in pairs.

The number of partitions of t as specified above is the coefficient of x^{t-3} in the series

$$1+(x^2+x^3)+(x^2+x^3)^2+\cdots\cdots\ ad\ inf.$$

i.e. $1/(1-x^2-x^3)$

If $\dfrac{1}{1-x^2-x^3}$ be expressed as the power-series

$$u_0+u_1x+\cdots\cdots+u_nx^n+\cdots\cdots$$

we have the recurrence relation

$$u_n=u_{n-2}+u_{n-3}\ (n>2)\ ;\ u_0=1,\ u_1=0,\ u_2=1.$$

If N_t denote the total number of ways of annihilation,

$$N_t=u_{t-3}+2u_{t-2}+u_{t-1}$$
$$=u\ +u_{t+1}$$
$$=u_{t+3}\qquad (t>2)$$

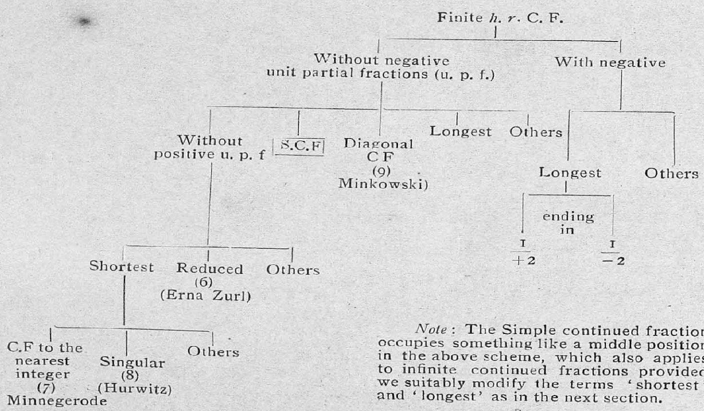
From direct considerations $N_1=1=u_4$ and $N_2=2=u_5$, so that

$$N_t=u_{t+3}\quad (t\geqslant 1) \qquad \ldots \quad (18)$$

The first twenty values of N_t are :

1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86
114, 151, 200, and 265. ... (19)

3·6 We may now conclude this section with a summary of various kinds of *h. r.* C. F's arranged as a genealogical tree :



Note : The Simple continued fraction occupies something like a middle position in the above scheme, which also applies to infinite continued fractions provided we suitably modify the terms 'shortest' and 'longest' as in the next section.

§ 4. 'Raschesten Kettenbruch entwicklung'

4·1 Following[5] Tietze's ideas, we introduce the terms 'more rapidly' 'more slowly', 'equally rapidly' converging in respect of two half-regular continued fraction developments of the same irrational number, say w, whose convergents are respectively:

$$(1)\quad P_1,\ P_2,\ \ldots\ P_n,\ \ldots$$

$$(2)\quad Q_1,\ Q_2,\ \ldots\ Q_n,\ \ldots$$

by the definition:

The first continued fraction is more rapidly (or more slowly) converging than the second, when for all positive integral n greater than a certain index m (a positive integer),

$$|P_n - \omega| < (or >)\ |Q_n - \omega|\ ;$$

otherwise, they are said to be equally rapidly convergent.

Let the S. C. F. development of 'ω' be

$$(a_0;\ a_1,\ a_2,\ \cdot\ \cdot\ \cdot\ );\qquad \ldots \qquad (a)$$

let the $h.\ r.$ C. F development obtained by using the extreme medial E_1-transformations to (a) be

$$\left(b_0;\ \frac{\varepsilon_v}{b_v}\right)_1^{\infty};\qquad \ldots \qquad (b)$$

and that due to the transformation c_1 applied to all batches of unit partial fractions in (a) be

$$\left(c_0:\ \frac{\varepsilon_v}{c_v}\right)_1^{\infty};\qquad \ldots \qquad (c)$$

and let $\left(d_0;\ \frac{E_v}{d_v}\right)^{\infty}$, be any other $h.\ r.$ C. F development of ω. (d)

Tietze has shown that (b) is the most slowly convergent and (c) one of the most rapidly convergent C. F's (raschesten Kettenbruch-entwicklung).

4·2. With a slight modification of Tietze's definition, we define the most rapidly convergent continued fraction developments of an irrational number ω by the following property:

If an infinite h. r. C. F, say (d), development of an irrational number ω has infinitely many convergents of the same order and value as the h. r. C. F, (c), development to the nearest integer, then (d) is one of the 'raschesten' variety.

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Let P_n, Q_n be the convergents of the nth order belonging to (d) and (c) respectively. If $P_n = Q_n$, it is evident that they correspond to

$$\left(d_0 ; \frac{\varepsilon_v}{d_v}\right)_1^n \text{ and } \left(c_0 ; \frac{\varepsilon_v}{c_v}\right)_1^n$$

which are both shortest *h. r.* C. F. developments, by § 3·4, of the same rational number; the first of these is therefore capable of being derived from the S. C. F, say, $F_t \equiv (a_0 ; a_1, a_2, \cdot \cdot a_t)$ by means of the P-transformation and annihilatory transformations; further P_n, Q_n are both principal convergents of (a).

Suppose there are infinitely many n's for which $P_n = Q_n$, then the transformations by which (d) is derived from (a) must consist only of the P- and (β)-transformations.

$$\text{If, now, } \left(d'_0 ; \frac{\varepsilon_v}{d'_v}\right)_1^\infty \quad . \quad . \quad . \quad . \qquad (d')$$

be the *h. r.* C. F. development derived from (a) by applying only the annihilatory transformations without the P-transformations, that are employed in obtaining (d), then the convergents of the nth order pertaining to (d) and (d') will coincide whenever they are both principal convergents of (a).

But, every convergent of (d') is a principal convergent of (a); so that if P'_n be the (n^{th}) convergent of (d') and $P_n = Q_n$, then $P'_n = P_n = Q_n$.

We have infinitely many such n's. Hence (d') is also of the 'raschesten' variety, by definition.

4·3. We can easily show that the annihilatory transformations employed in deriving (d') from (a) are just of the type c_1, c_2, or $c_1 c_2^e$ applied to even batches of consecutive unit partial quotients and c_1 to other batches, including, if any, the infinite one. For, otherwise, by taking n sufficiently large, we will have the shortest continued fraction development $\left(d'_0 ; \frac{\varepsilon_v}{d'_v}\right)_1^n$ obtained by applying at least to one batch of successive unit partial fractions of the S. C. F., F_t say, an eclectic transformation different from the appropriate one; which is impossible by (10). We may note in this connection, that, if $a_t = 1$, it is either the last partial quotient of a batch of an even number of unit partial quotients in (a) or a medial partial quotient occupying an odd rank in a batch of unit partial quotients. If (a) should end in an infinite number of unit partial fractions which are modified in

(d') by a transformation other than c_1, then we may take n sufficiently large so that the shortest $h.\ r.$ C. F. development $\left(d'_0;\ \frac{\varepsilon_v}{d'_v}\right)_1^n$ is obtained from F_t where $a_t=1$ by annihilatory transformations the last of which is different from C_1 and therefore not appropriate to the last batch of unit partial fractions in F_t, which is necessarily odd. This is again a contradiction to (11).

Again, by (11), the residues of (d') are all absolutely less than $\frac{1}{2}(\sqrt{5}-1)$; and by Lemma II, a P-transformation applied to $\frac{1}{+2}$ which is followed by a positive residue less than $\frac{1}{2}(\sqrt{5}-1)$ does not introduce any residue absolutely greater than or equal to $\frac{1}{2}(\sqrt{5}-1)$. Hence the residues of (d) are also all absolutely less than $\frac{1}{2}(\sqrt{5}-1)$. This proves Tietze's[5] famous theorem, which may be quoted here in the original:

'Die raschesten halbregelmässigen Entwicklung einer zahl sind jene halbregelmässigen Entwicklung, bei denen jeder Entwicklungs rest γ_{v+1} der Bedingung genugt

$$|\gamma_{v+1}| < \tfrac{1}{2}(\sqrt{5}-1) = 0{\cdot}6180339887 \cdots (r=0,\ 1,\ \cdots)\text{'}$$

Tietze's theorem itself follows from the observations in § 2·5 and the properties of convergents mentioned in § 1·12 (iii) and § 1·24.

We may be permitted to conclude here with the remark that we have been able to give a simple and unified view of the half-regular continued fraction by a detailed examination of the properties of the *two* fundamental transformations (α) and (β) and the *ten* derived ones,

viz. $E_1, E_2, E_3, P, C_1, C_2, C_1C_2 \cdots C_2^e, C_1C_2 \cdots C_2^0,$
$C_2C_2 \cdots C^e_2$, and $C_2C_2 \cdots C_2^0$.

References

(1) O. PERRON: *Die Lehre von den Kettenbrüchen,* 1929, pp. 149-190.

(2) J. L. LAGRANGE: *Oeuvres* VII.

(3) K. TH. VAHLEN: Über Näherungswerte und Kettenbrüche. *J. f. Math.* 115, 1895.

(4) G. CHRYSTAL: *Algebra,* Part II, 1919; pp. 494, 495.

(5) H. TIETZE: Über die raschesten Kettenbruchentwicklungen reeller Zahlen. *Monatshefte für Mathematik und Physik,* XXIV, 1913.

(6) ERNA. ZURL: Theorie der reduziert-regelmässigen Kettenbrüche, *Mathematische Annalen* 1935.

(7) B. MINNIGERODE: Über eine neue Methode, die Pellsche Gleichung aufzulösen. *Gött. Nachr.* 1873.

(8) A. HURWITZ: Über eine besondere Artder Kettenbruchentwicklung reeller Grössen. *A. Math.* 12, 1889.

(9) H. MINKOWSKI: Über die Annäherung an eine reelle Grösse durch rationale zahlen. *Mathematische Annalen* 54, 1901.

ON WONDERFUL DEMLO NUMBERS

BY

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1. The object of this paper is to discuss the rationale of certain curious results in the multiplication of numbers written in the usual scale of ten, an example of which is the following :—

$$1234567654321 \times 243 = 299999940000003.$$

I call this a "*partition and insertion property*" since, when the multiplier 243 is partitioned into digits and repeated digits 9 and 0 are inserted we get the product. I show in this paper how to obtain in other numbers with this property.

Notation and Definitions

2. To save space I shall write $(a)^n$ to indicate the digit a repeated n times. Thus $(1)^2\ 0\ 2(9)^3 = 1102999$.

I use the word "Demlo Number" to denote an integer which may be partitioned into three groups of which the middle group is of the form $(a)^l$ and the sum of the numbers in the first and last groups (supposed to have the *same* number of digits) is of the form $(a)^m$. Thus 12488764 is a Demlo Number of which the middle part is 88 and whose extreme groups 124 and 764 add up to 888.

A special class of Demlo Numbers are "Wonderful Demlo Numbers" which are defined as follows :—

Def : A wonderful demlo number is the square of an integer of the form $(1)^l$ where $1 < l \leqslant 9$. Thus

$$121 ;\ 12321 ;\ 1234321 ;\ \text{etc. and } \{1^{(9)}\}^2 = 13345678987654321$$

are all wonderful demlo numbers. Here the middle part consists of a single digit while the first and last groups consist of numbers in rising or falling order, I shall use W to denote a wonderful demlo number.

Def : A Partition number is an integer of the form $9 \times ab$ where $b \geqslant (a+1)$ and $a + b \leqslant 9$. I shall use the letter P to denote a "partition number."

Example :—9×45 is a partition number.

The Partition and insertion property

3. LEMMA I: If ABC be the successive digits of a partition number, then $A=a$, $B=b-a-1$, $C=10-b$.

For $ABC = ab \times 9 = ab(10-1) = ab0 - ab = a(b-a-1)(10-b)$.

Thus $35 \times 9 = 315 = 3(5-3-1)(10-5)$.

LEMMA II: If W be the square of $(1)^l$, then

$$W \times 9 = (1)^l\,(0)^l - (1)^l.$$

For, $W \times 9 = \{(1)^l\}^2 \times 9 = \left\{\frac{(10)^l - 1}{9}\right\}^2 \times 9$

$$= \frac{(10)^{2l} - 2(10)^l + 1}{9}$$

$$= (10)^l \left\{\frac{(10)^l - 1}{9}\right\} - \frac{(10)^l - 1}{9} = (1)^l\,(0)^l - (1)^l.$$

Example: $12321 \times 9 = 111000 - 111$.

THEOREM I: $ABC \times W = A(c)^{l-1}B\,(9-c)^{l-1}C$

where $P = ABC$ and $c = a+b$.

$$\begin{aligned} \text{For } W \times ABC &= W \times P = W \times 9 \times ab \\ &= \{(1)^l\,(0)^l - (1)^l\} \times ab \text{ by Lemma II} \\ &= a(c)^{l-1}\,b(0)^l - a(c)^{l-1}b \\ &= a(c)^{l-1}\,(b-a-1)\,(9-c)^{l-1}\,(10-b) \\ &= A(c)^{l-1}B\,(9-c)^{l-1}C \text{ from Lemma I} \end{aligned}$$

Thus $1234567654321 \times 243 = 299999940000003$ (Here $243 = 9 \times 27$, so $c = a+b = 2+7 = 9$ and $9-c=0$ and both these are repeated.)

Corollary I: $\{(3)^l\}^2 = (1)^{l-1}\,0(8)^{l-1}\,9$

For $\{(3)^l\}^2 = \{(1)^l\}^2 \times 9 = W \times P$.

Thus the square of 3333 is 11108889.

Corollary II: $(3)^l\,(6)^l = (2)^{l-1}1\,(7)^{l-1}8$.

Corollary III: $\{(6)^l\}^2 = (4)^{l-1}3\,(5)^{l-1}6$.

Corollary IV: $\{(9)^l\}^2 = (9)^{l-1}8\,(0)^{l-1}1$.

THEOREM III: There are only 25 partition numbers P altogether and they are: 9, 18, 27, 36, 45, 54, 63, 72, 81, 108, 117, 126, 135, 144, 153, 162, 207, 216, 225, 234, 243, 306, 315, 324 and 405.

For, from the definition $P = 9 \times ab$, where $a + b = 9$ and $b \geqslant a + 1$ Hence a and b can assume only the following values:

a	b
0	1, 2, 3, 4, 5, 6, 7, 8, 9
1	2, 3, 4, 5, 6, 7, 8
2	3, 4, 5, 6, 7
3	4, 5, 6
4	5

Examples to Illustrate Theorem III

$$123454321 \times 216 = 26666133336$$

Here the digits 2, 1, 6 are partitioned and 3 and 6 are inserted in the gaps. Similarly

$$123454321 \times 315 = 38888111115$$

Again

$$12345654321 \times 108 = 1333330666668.$$

Progress in the History of Mathematics During the Last Forty Years

In 1898 the first number of volume 1, entitled "Grundlagen der Arithmetik" of the now widely used *Encyklopädie der mathematischen Wissenschaften* was published. On page 12 thereof there appears a note (18) on the negative numbers which involves various statements which are now commonly known to be incorrect and hence they exhibit decided progress in the history of our subject during the last forty years as it may be assumed that such a standard work in its day represents about the best that was then available along the lines treated. One of these statements is that the first traces of negative numbers appear in the work of the Indian mathematician Bhaskara (born in 1114) who distinguished between the positive and the negative value of the square roots of a positive number. On the contrary, it is now known that negative numbers were used much earlier in India and that the writings of the ancient Babylonians already contain such common rules in multiplication as $+ - = - + = -$, and $- - = +$. Cf. *Zentralblatt für Mathematik,* volume 16, page 385 (1937).

The noted incorrect assertion in regard to the first traces of negative numbers is followed in the same note by the assertion that the Arabs recognized negative roots of an equation which is equally commonly known now to be incorrect. In particular, J. Tropfke observes in his widely known *Geschichte der Elementar-Mathematik,* volume 2, page 97 (1933), that negative solutions were not allowed by the Arabians as may be seen, among other places, in the text-book on algebra by Alkarhi, about 1010. Minor misstatements in the same note are that the *Ars Magna* of Cardan appeared in 1550 instead of in 1545, and that T. Harriot, about 1600, considered negative numbers by themselves and allowed them to constitute a member of an equation. It is now known that the Babylonians already allowed a negative number to appear by itself as a member of an equation. Cf. O. Neugebaner, *Mathematische Keilschrift Texte,* part 3, page 38 (1937).

The same note terminates with two other incorrect statements in the same sentence. The former of these is to the effect that the real calculating with negative numbers begins with R. Descartes, who died in 1650. As a matter of fact, this statement is directly

contradicted in the corresponding note (149) of the French edition of the same encyclopedia, page 35, where it is asserted that the systematic calculating with negative numbers was posterior to R. Descartes. The second misstatement in the same sentence to which we referred is that R. Descartes assigned to the same letter sometimes a positive and sometimes a negative value. As a matter of fact, a letter had only a positive value in R. Descartes's writings. It should, however, be noted in this connection that by placing a point before a letter R. Descartes indicated that the latter could have both a positive and also a negative value. Cf. J. Tropfke, *Geschichte der Elementar-Mathematik*, volume 2, page 100 (1933).

Although there are fewer actual errors in note 24 of the article cited at the beginning of the present discussion than in note 18 thereof the former involves a statement which is probably more surprising to many readers than any one which appears in the latter, viz., that the positional number system of the Babylonian astronomers to the base 60 involved the use of 59 different numerals. This would correspond to the 9 different digits of our common number system to the base 10, but there seems to be no evidence that it ever existed. It is now well known that the Babylonian number system involves the use of the base 10 up to 60 so that it was unnecessary to have 59 different numerals. In fact the symbols for unity and for 10 are the only ones which were commonly used. Cf. O. Neugebaner, *Vorlesungen über Geschichte der Antiken Mathematischen Wissenschaften*, volume 1, page 4 (1934).

The two opening sentences of note 24 are to the effect that decimal fractions originated during the 16th century and that John Kepler (1571-1630) introduced the decimal comma. It is now known that a decimal comma was used by John Napier in his *Rabdologia* of 1617 and that this is the first known use thereof. The decimal period was used a little earlier by C. Clavius in his *Astrolabium* of 1593. Simon Stevin (1540-1620) was the first to treat decimal fractions systematically and extensively, and he has often been credited with their invention. He devoted a small pamphlet to this subject which appeared in 1585 both separately and as an appendix to his *Practique d'Arithmetique* and he urged the governments to introduce decimal money, weights and measures in order to simplify calculations. In fact, he also pointed out the importance of dividing the angle decimally and thus became the most influential forerunner of the use of decimal systems. Special decimal fractions appeared before the beginning of the 16th century in Arabia.

In note 25, page 22, it is stated that traces of calculating with powers and roots appear already in the 14th to the 16th century. It is now known that such traces appear already in the works of Archimedes and possibly earlier in the work of the ancient Babylonians as was noted by O. Neugebaner on pages 200 and 201 of the history cited at the close of next to the preceding paragraph. It is not implied that all of the historical notes in the *Encyklopädie der mathematischen Wissenschaften* have become unreliable in the comparatively short time since this work has been before the public. A considerable number of historical statements represent advances in this history of permanent value but there are others which have become antiquated in recent years as a result of the rapid recent progress along historical lines. It may be of interest to emphasize a few definite evidences of this progress by means of a publication which is in general still very favourably known in order to exhibit more clearly the fundamental nature of this progress.

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On a Property of Self-Sufficient Sets

1. A set (A) of n distinct non-zero positive integers: $a_1, a_2, a_3 \ldots\ldots a_n$; where each $a<p$, p being an odd prime, is said to be a self-sufficient p-set, when the product of every two (or more) members of the set is congruent modulo p, with a member of the set itself; i.e. when $a_i\ a_j \equiv a_m \pmod{p}$, $0<(i,\ j,\ m)\leqslant n$.

Since any $a^{p-1}\equiv 1 \pmod{p}$, unity must be a member of such a set.

2. The object of this note is to prove the following

THEOREM*: *The n members of any self-sufficient p-set (A), are the k-ic residues of p, where $kn=p-1$.*

Let g be a primitive root of p. Then corresponding to the given set (A), there is another set (B) of positive integers: $b_1, b_2, b_3, \ldots\ldots b_n$; given by the relation: $g^{b_r}\equiv a_r \pmod{p}$, $1\leqslant r\leqslant n$; and $0<b_r\leqslant\phi(p)$.

Evidently the sum of every two (or more) members of (B) is congruent modulo $\phi(p)$ with a member of (B) itself; i.e. $b_i+b\equiv b_m$ (mod $\phi(p)$).

Let k denote the greatest common divisor of the members of (B).

* This theorem was suggested to me by Dr. S. Chowla.

Then the numbers: k, $2k$, $3k$, . . . , $(p-1)$; can be represented modulo $\phi(p)$ as the sums of b's: and no other non-zero positive number $\leqslant \phi(p)$ can be so represented. Hence the members of (B) are k, $2k$, $3k$, $4k$, . . . , $(p-1)$; in some order.

Let $b_r = rk$, so that $b_n = nk = (p-1)$.

Then $a_r \equiv g^{rk}$ (mod. p).

If $g^r \equiv t$ (mod. p), we get $a_r \equiv t^k$ (mod. p).

Since there are just n k-ic residues of p and no more, the numbers of set (A) are all the k-ic residues of p.

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The Twisted Straight Line

1. According to a well-known theorem in differential geometry, there exists a curve, unique except for position and orientation in space, whose curvature k and torsion τ are assigned functions

$$\kappa = f(s); \qquad \tau = \phi(s)$$

of the arc length s. When $\phi(s) \equiv 0$, we have a plane curve with curvature $f(s)$. I am not sure whether the other case, when $f(s) \equiv 0$ is sufficiently well known among students and teachers. The curve is now *a twisted straight line, i.e.* a straight line with a definite osculating plane and hence also a definite torsion $\phi(s)$ associated with each point. By changing the function $\phi(s)$ we vary the twist of the straight line. A rope which is being twisted under tension is a material symbol of a twisted straight line of nearly constant torsion.

One immediate application of this idea is that every conoidal surface may be considered as the surface of principal normals or surface of binormals of a twisted straight line. Now it is known that the sum of the principal curvatures at a point P of the surface of binormals is given by

$$\mathrm{J} = \frac{\kappa^2 + \kappa\tau^2 u^2 - \tau' u}{(1+\tau^2 u^2)^{3/2}} \qquad \ldots \quad (1)^*$$

where u is the distance of P from the curve measured along the binormal through it. Putting $k=0$ we have $\mathrm{J} = -\tau' u/(1+\tau^2 u^2)^{3/2}$. This vanishes when the straight line has constant torsion and the binormal surface is in this case a helicoid. Hence the helicoid is a minimal surface.

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* Weatherburn: *Differential geometry of three dimensions,* Vol. I, (1931) P. 72.

Note on an Inequality

1. The object of this note is to generalise an inequality on a definite integral given in problem 121 in Polya-Szegö: *Aufgaben u. Lehrsatze,* II Abschn, *p.* 62. The problem there given is equivalent to the following :—

Let $f(a)=f(b)=0, \quad |f'(x)|\leqslant M$ in $a\leqslant x\leqslant b$.

Then $$\frac{M(b-a)^2}{4} > \left| \int_a^b f(t)\,dt \right|$$

We propose proving the following generalisation :

If M is the upper bound of $|f'(x)|$ *in* (a, b), *then*

$$|\mathbf{J}| \equiv \left| \int_a^b f(t)\,dt - \frac{1}{2}(b-a)\left\{f(a)+f(b)\right\} \right| < \frac{M(b-a)^2}{4} - \frac{1}{4M}\left\{f(b)-f(a)\right\}^2 \quad \cdots\cdots \quad \text{(E)}$$

Before proceeding to the proof of the inequality (E), it may be remarked that

$$|\mathbf{J}| = \left| \int_a^b \left(\frac{a+b}{2} - x\right) f'(x)\,dx \right| \leqslant \frac{M(b-a)^2}{4}$$

which is obviously a much weaker inequality than (E).

2. *Proof of (E)*: We have

$$\int_a^b f(t)dt - \tfrac{1}{2}(b-a)\{f(b)+f(a)\} = \int_a^b \left\{ f(t) - \frac{f(b)+f(a)}{2} \right\} dt$$

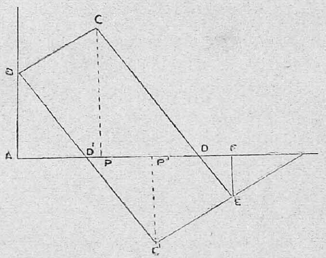
$$= \int_a^b \phi(t)\,dt, \text{ say}$$

where $\phi(t)=f(t)-\frac{1}{2}\{f(b)+f(a)\}$, so that $\phi(a)+\phi(b)=0$.

Let $\alpha=\phi(a)=-\frac{1}{2}\{f(b)-f(a)\}$. Then the curve $y=\phi(t)$ lies below the lines

$y=\alpha+M(t-a)$ (Line BC)

$y=-\alpha-M(t-b)$, (Line CE)



i.e. it lies below the polygonal line BCDE. Similarly it lies above the polygonal line BD′C′E. Hence

$$\int_a^b \phi(t)\,dt \leqslant \text{area under BCDE}$$
$$= \tfrac{1}{2}(\mathrm{CP}+\mathrm{AB})\mathrm{AP}+\tfrac{1}{2}(\mathrm{CP}-\mathrm{EF})\mathrm{PF}$$

An easy calculation gives

$$\mathrm{CP}=\tfrac{1}{2}\mathrm{M}(b-a),\ \mathrm{EF}=\mathrm{AB}=\alpha,\ \mathrm{AP}=\tfrac{1}{2}(b-a)-\frac{\alpha}{\mathrm{M}},\ \mathrm{PF}=\tfrac{1}{2}(b-a)+\frac{\alpha}{\mathrm{M}}.$$

$$\therefore \int_a^b \phi(t)\,dt \leqslant \frac{1}{2}\Big[\Big\{\frac{1}{2}\mathrm{M}(b-a)+\alpha\Big\}\Big(\frac{b-a}{2}-\frac{\alpha}{\mathrm{M}}\Big)$$
$$+\Big\{\frac{1}{2}(b-a)\mathrm{M}-\alpha\Big\}\Big(\frac{b-a}{2}+\frac{\alpha}{\mathrm{M}}\Big)\Big]$$
$$=\frac{\mathrm{M}(b-a)^2}{4}-\frac{\alpha^2}{\mathrm{M}}=\frac{\mathrm{M}(b-a)^2}{4}-\frac{1}{4\mathrm{M}}\Big\{f(b)-f(a)\Big\}^2$$

Similarly, considering the curve BD′C′E we get

$$\int_a^b \phi(t)\,dt \geqslant \frac{\alpha^2}{\mathrm{M}}-\frac{\mathrm{M}(b-a)^2}{4}$$

Hence

$$\Big|\int_a^b \phi(t)\,dt\Big| = \Big|\int_a^b f(t)\,dt-\tfrac{1}{2}(b-a)\{(f(a)+f(b)\}\Big|$$
$$\leqslant \frac{\mathrm{M}(b-a)^2}{4}-\frac{1}{4\mathrm{M}}\Big\{f(b)-f(a)\Big\}^2.$$

It is quite clear that equality can only occur when $\phi(t)$ coincides completely with BCDE or BD′C′E, which is however not possible since $\phi'(t)$ exists at the points P, P′. Hence (E) is proved.

Since we can approximate to BCDE as closely as we like by a curve through B, E, it is obvious that the inequality cannot be improved.

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A Note on the Zeros of $\sum_{r=0}^{n} \frac{z^r}{r!} = 0.$

In this note it is proposed to shew, by an elementary method, that the zeros of $\sum_{r=0}^{n} \frac{z^r}{r!}$, for large n, lie in the region $n > |z| > n/e^2$.

1. We shall first prove that for all n, the zeros all lie in $|z| < n$.

Let $f_n(z) = \sum_{r=0}^{n} \frac{z^r}{r!}$, and $\phi(z) = f_n(nz)$

Let
$$g(z) = z^n \phi\left(\frac{1}{z}\right) = b_0 + b_1 z + \cdots\cdots + b_n z^n;$$

then
$$b_0 = \frac{n^n}{n!},\ b_1 = \frac{n^{n-1}}{(n-1)!}, \cdots\cdots\cdots,$$
and, obviously, $b_0 = b_1 > \cdots\cdots\cdots\cdots\cdots > b_n$.

Then all the zeros of $g(z)$ lie outside the unit-circle (Vide e.g. Landau: *Ergebnisse der Funktionentheorie*, p. 26, Satze von Eneström).

As the proof is short and simple we give it here:
$$(1-z)g(z) = b_0 - \{(b_0 - b_1)z + (b_1 - b_2)z^2 + \cdots + (b_{n-1} - b_n)z^n + b_n z^{n+1}\}$$

Hence for $|z| \leqslant 1$, excluding $z = 1$, we have
$$|(1-z)g(z)| \geqslant b_0 - \{(b_0 - b_1) + (b_1 - b_2) + \cdots + (b_{n-1} - b_n) + b_n\} = 0.$$

It is obvious that the right side in the above expression for $(1-z)g(z)$ can be zero only when $z = 1$. Hence $|g(z)| > 0$ when $|z| \leqslant 1$.

Going back now to the function $f_n(z)$, we see immediately that all its roots lie in the region $|z| < n$.

2. At a zero, say z_0, of $f_n(z)$ we have
$$\frac{z_0^{n+1}}{(n+1)!} + \frac{z_0^{n+2}}{(n+2)!} + \cdots\cdots = e^{z_0} \cdots\cdots \quad (2\cdot1)$$

Also $|z_0| < n$. Let $|z_0| = (n+1)u_0$

Then
$$\exp\{-|z_0|\} < \frac{|z_0|^{n+1}}{(n+1)!} + \cdots\cdots < \frac{|z_0|^{n+1}}{(n+1)!\ (1-u_0)}$$
$$= \frac{(\overline{n+1} \cdot u_0)^{n+1}}{(\overline{n+1}/e)^{n+1}} \cdot \frac{1+\varepsilon_n}{\sqrt{(2\pi n)} \cdot (1-u_0)} \quad \text{for large } n.$$

Hence
$$u_0 \cdot e \cdot (1+\varepsilon_n)^{1/(n+1)} \div \{(1-u_0)\sqrt{2\pi n}\}^{1/(n+1)} > e^{-u_0} \cdots\cdots \quad (2\cdot2)$$

Now by (§ 1) $u_0 < n/(n+1,)$ so that $1 < \frac{1}{(1-u_0)} < n+1$
and $1 < (1-u_0)^{-1/(n+1)} < (n+1)^{1/(n+1)} = 1 + \varepsilon'_n$,

Hence from (2·2) we have

$$u_0 e^{u_0+1} > 1 + \varepsilon''_n$$

or $1 + u_0 + \log u_0 > \varepsilon'''_n$.

Now $1 + u_0 + \log u_0$ is a monotonic function which vanishes for one value u_0 in $\frac{1}{e} > u_0 > \frac{1}{e^2}$

Hence when n is large $u_0 > \frac{1}{e^2}$ whence $n > |z_0| > \frac{n}{e^2}$ where z_0 is any root of $\sum_{r=0}^{n} \frac{z^r}{r!} = 0$.

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Inequalities Connected with a Tetrahedron

In this note are proved certain inequalities connected with a tetrahedron. Most of them are analogous to the inequalities relating to a triangle.

1. Let ABCD be a tetrahedron the areas of whose faces are $\alpha, \beta, \gamma, \delta$ and whose volume is V. Let R, r be the circum- and in-radii, respectively, of ABCD and let S be the total surface of the tetrahedron. If a, a'; b, b'; c, c'; be the lengths of the three pairs of opposite edges of ABCD, we have

$$V = \tfrac{1}{6}\, aa'd_1 \sin \theta_1 \leqslant \tfrac{1}{6}\, aa'b$$

where d_1, θ_1, are respectively the shortest distance and the angle between the pair of opposite edges of lengths a, a'.

Similarly $V < \frac{1}{6}\, cc'\, b'$.

From these two we have $V < \frac{1}{6}\, (aa'\ bb'\ cc')^{\frac{1}{2}}$.........(1·1)

O being the circum-centre of ABCD, if v_1, v_2, v_3, v_4 be the volumes of the four tetrahedra OBCD, OCAD, OABD, OABC,

we have, by (1·1), $v_1 < \frac{R^{3/2}}{6} \sqrt{(a\, b'\, c')} < \frac{R^{3/2}}{6} \sqrt{(4\, R\, \alpha)}$.

$$i.e.\ v_1 < \frac{R^2}{3} \sqrt{\alpha}.$$

and similar inequalities hold for v_2, v_3 and v_4. Hence, by adding up, we obtain

$$V < \frac{R^2}{3} (\sqrt{\alpha} + \sqrt{\beta} + \sqrt{\gamma} + \sqrt{\delta}) \qquad \text{... (1·2)}$$

Since, of all triangles, inscribed in a circle, the equilateral triangle has the maximum area and since the circumradius of any tetrahedron is greater than or equal to that of any of its faces, we have

$$R^2 \geqslant \frac{4A}{3\sqrt{3}} \quad \ldots (1\cdot3)$$

where A is the area of the greatest of the faces of the tetrahedron.

2. We next proceed to obtain an inequality connecting the in- and circum-radii of a tetrahedron. It is well known that of al- tetrahedra inscribed in a sphere of radius R the one whose volume as also the total surface area is a maximum is the regular tetrahedron and it is easily found that the value of this maximum surface area is $8R^2/\sqrt{3}$. Hence this theorem gives for any tetrahedron, the inequality $R^2 \geqslant \frac{1}{8} S\sqrt{3}$ ... (2·1)

The inequality becomes an equality when and only when the tetrahedron is regular. Again from a theorem due to Steinitz which states that "of all tetrahedra of given surface area, the regular tetrahedron has the maximum volume and hence the maximum in-radius" we obtain for any tetrahedron,

$$r^2 \leqslant \frac{S}{24\sqrt{3}} \quad \ldots (2\cdot2)$$

the equality holding when and only when the tetrahedron is regular. From (2·1) and (2·2) we obtain $r/R \leqslant 1/3$ where, as before the equality holds only when the tetrahedron is regular. The above result is the three-dimensional analogue of the result "the in-radius of a triangle is less than or equal to half of its circum-radius."

... (2·3)

Since the volume V of a tetrahedron is given by the formula $V = rS/3$, we obtain, from (2·2)

$$S^3 - 216\sqrt{3}\,V^2 \geqslant 0. \quad \ldots (2\cdot4)$$

This is the three-dimensional analogue of the inequality

$$S^2 - 12\sqrt{3}A \geqslant 0 \quad \ldots (2\cdot5)$$

relating to the perimeter S and the area A of a triangle. It may, perhaps, be useful to observe, here, that the inequalities (2·4) and (2·5) are sharper than those given by Schwarz for a convex solid and a convex oval † viz.

$$S^3 - 36\pi\, V^2 \geqslant 0 \text{ and } S^2 - 4\pi A \geqslant 0.$$

† Blaschke: *Kreis and Kugel* Pp. 30-31 and Pp. 78-79.

Annamalainagar.

B. R. VENKATARAMAN,
Research Student.
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ASTRONOMICAL NOTES

Planet Notes from October to December 1938

(All times mentioned here are I. S. T. and the co-ordinates of planets given are for 5 hours 30 minutes I. S. T.)

Sun. The Sun after passing the equator on 23rd September moves with an increasing South declination and attains its maximum declination South of the equator on the evening of 22nd December thus marking the beginning of winter in the Northern hemisphere. During the period October to December the Sun will be passing from the centre of Virgo to the centre of Sagittarius. Its Right Ascension will be changing from 12 h. 26 m. to 18 h. 38 m. and Declination from 2° 50′ S. to its maximum 23° 27′ S on 22 December and then decreases to 23° 10′ S by the end of December. On 22nd November there will be a partial eclipse of the Sun invisible in India.

The Sun's position at the beginning of each month is as follows :—

		1st October	*1st November*	*1st December*
R. A.	...	12 h 26 m	14 h 22 m	16 h 26 m
Dec.	...	2° 51′ S	14° 9′ S	21° 40′S

Moon. The following phases of the Moon will occur during the period October to December.

	New Moon	*1st Quarter*	*Full Moon*	*Last Quarter*
October	23rd at 2-12 P.M.	1st at 5-15 P.M. 31st at 1-15 P.M.	9th at 3-7 P.M.	16 at 2-54 P.M.
November	22nd at 5-35 A.M.	30th at 9-29 A.M.	8th at 3-53 A.M.	14 at 9-50 P.M.
December	21st at 11-37 P.M.	30th at 4-23 A.M.	7th at 3-52 P.M.	14 at 6-47 A.M.

The Moon will be nearest the earth on 16th October, 11th November, and 9th December and will be farthest on 2nd, 30th October, 27th November and 24th December. On 8th November there will be a total eclipse of the Moon visible in India. The circumstances of the eclipse will be as follows :—

Beginning of total eclipse	...	3-15 A.M.	I. S. T.
Middle ,,	...	3-56 A.M.	,,
End ,,	...	4-38 A.M.	,,

Moon's position at the beginning of each month :—

		1st October	*1st November*	*1st December*
R. A.	...	18 h 8 m	21 h 5 m	23 h 7 m
Dec.	...	19° 53′ S	11° 12′ S	0° 44′ S

Mercury. The planet with direct motion moves from Virgo to Sagittarius and reaching the stationary point in the constellation on December 24th retraces the path and enters Scorpio on 13th December and remains in it till the end of December. During October Mercury is too near the Sun to be visible but by the end of November it can be seen near the Western horizon for about an hour and a half after sunset. Again by the beginning of December it approaches the Sun and by the end of the month the planet is visible as a morning object rising in the east an hour and a half before the Sun.

The following phenomena of Mercury will take place during this period :—

Mercury stationary	...	4th and 24th December
Superior conjunction with Sun	...	10th October
Inferior ,,	...	14th December
Conjunction with Venus	...	9th November
Conjunction with Moon	...	24th November & 20th December
Greatest elongation 22° E	...	25th November.

The position of the planet is as follows :—

		1st October	*1st November*	*1st December*
R. A.	...	12 h 2 m	15 h 13 m	17 h 54 m
Dec.	...	1° 46′ N	19° 17′ S	25° 20′ S

Venus. During the beginning of October Venus enters Scorpio and by the end of November it moves to Libra and again moves to Scorpio by the end of December. Venus is an evening object during October setting earlier day by day till it nears the Sun on November when it cannot be seen, but during December it is visible as a morning object. Like Mercury this planet is stationary on two occasions i.e., on 30th October and 9th December and is in inferior conjunction with the Sun on 20th November. Venus attains its greatest brilliancy on 16th October and 26th December. It is in conjunction with the Moon on 26th October and 18th December.

The positions of the planet will be as follows :

		1st October	*1st November*	*1st December*
R. A.	...	15 h 11 m	16 h 7 m	15 h 16 m
Dec.	...	22° 39′ S	26° 54′ S	18° 23′ S

Mars. This planet moves with a direct motion from Leo to Libra during this period. Mars is easily visible as a morning object on the eastern sky during the early morning hours. Mars is in conjunction with the Moon on 21st October, 18th November and 17th December. The planet is also in conjunction with the planet Neptune on 12th October.

The positions of the planet will be as follows:—

	1st October	*1st November*	*1st December*
R. A. ...	11 h 5 m	12 h 17 m	13 h 27 m
Dec. ...	7° 8′N	0° 38′S	7° 59′S

Jupiter. The planet with a retrograde motion enters Capricornus from Aquarius on 2nd October and remaining in this constellation for about a month retraces its path with direct motion and again moves to Aquarius on the 4th November. Jupiter is visible throughout this period as an evening object. Jupiter is in conjunction with the Moon on the following dates 5th Oct., 2nd Nov., 29th Nov. and 27th Dec. and is stationary on the 19th October.

The positions of the planet will be as follows:—

	1st Oct.	*1st Nov.*	*1st Dec.*
R. A. ...	21 h 43 m	21 h 42 m	21 h 52 m
Dec. ...	15° 1′ S	15° 2′ S	14° 5′ S

Saturn. This planet is visible throughout the night during October and during the latter part of December it will be visible till midnight since it sets by midnight at the end of December. Saturn will be in Pisces throughout. Saturn is in opposition with the Sun on 8th October and in conjunction with the Moon on 9th October, 5th November, 3rd and 30th December. The planet is stationary on 15th December. The positions of the planet will be as follows:—

	1st Oct.	*1st Nov.*	*1st Dec.*
R. A. ...	1 h 0 m	0 h 52 m	0 h 46 m
Dec. ...	3° 27′ N	2° 34′ N	2° 5′ N

Uranus. This planet will be visible shortly after 8 P.M. during October and by the end of December it will be visible only till 2 hours after midnight. On 8th Nov. Uranus will be in opposition with the Sun.

The positions of Uranus will be as follows:—

	1st October	*1st November*	*1st December*
R. A. ...	3 h 0 m	2 h 55 m	2 h 50 m
Dec. ...	16° 37′N	16° 18′ N	15° 57′ N

Neptune. This planet will be visible only during the morning hours. It is stationary on the 26th December.

	1st October	*1st November*	*1st December*
R. A. ...	11 h 31 m	11 h 34 m	11 h 37 m
Dec. ...	4° 23′ N	4° 0′N	3° 46′N

H. SUBRAMANI AIYER.

SOLUTIONS TO QUESTIONS

Question 1607

(R. RAMANUJAM) :—Prove the theorem of Pythogoras without constructing squares on the sides of the triangle.

Solution by A. Ranganatha Rao.

Let ABC be a triangle, right-angled at A. Draw AF perpendicular to BC and produce it to D such that AD = BC. Draw DE perpendicular to AC.

The two right-angled triangles BAC, AED are congruent.

$$\therefore \quad CA = DE \text{ and } AB = AE.$$

$$\text{Quad. } ABDC = \triangle ABD + \triangle ADC.$$
$$= \tfrac{1}{2}\, AC\cdot DE + \tfrac{1}{2} AB\cdot DE$$
$$= \tfrac{1}{2}\,(AC^2 + AB^2).$$

$$\text{Also quad. } ABDC = \tfrac{1}{2}\, BC\cdot AD = \tfrac{1}{2}\, BC^2.$$

$$\text{Hence} \quad BC^2 = AB^2 + AC^2.$$

The well-known Bhaskara-Wallis' solution by the use of similar triangles FBA and FAC is given by

T. Manivanan, T. K. Ragavachari and the proposer.

Question 1745

(D. R. KAPREKAR) :—We have 91 × 819 = 74529 and also 9901 × 980199 = 9704950299 where the corresponding numbers in the second multiplication are obtained by inserting 9 and 0 alternately before the first digit and between the successive digits of all the numbers in the first multiplication. It will be found that the multiplication is still true if we insert alternately blocks of 99 and 00 or 999 and 000 and so on.

It is required to find the principle underlying the result and to find other number pairs like 91 and 819 with the same property.

Solution by V. Narasimhamurti.

Let $10-a_1, a_2, 10-a_3, \cdots$ and $10-b_1, b_2, 10-b_3, \cdots$ be the digits in order from the left of the two given numbers of m and n digits. Hence the given numbers are

$$\{(10-a_1)\,10^{m-1} + a_2\,10^{m-2} + \cdots\} \text{ and } \{(10-b_1)\,10^{n-1} + b_2\,10^{n-2} + \cdots\}$$

Let their product be the number of l digits

$$(10-c_1)\,10^{l-1}+c_2\,10^{l-2}+(10-c_3)\,10^{l-3}+\cdots\cdots$$

where $10-c_1, c_2, \cdots\cdot$ are the digits from the left in the product. A general statement of the problem would require that the following equality will hold for every integer $s>0$, viz.,

$$\{(10^s-b_1)(10^s)^{n-1}+b_2(10^s)^{n-2}+\cdots\cdot\}\{(10^s-a_1)(10^s)^{m-1}+a_2(10^s)^{m-2}+\cdots\}$$
$$=(10^s-c_1)\,(10^s)^{l-1}+c_2\,(10^s)^{l-2}+\cdots\cdot \qquad \ldots \quad \text{I}$$

Putting $10^s=x$, I can be written in the form

$$\{x^m-a_1\,x^{m-1}+(a_2+1)x^{m-2}-+\}\;\{x^n-b_1\,x^{n-1}+(b_2+1)x^{n-2}-+\}$$
$$=\{x^l-c_1\,x^{l-1}+(c_2+1)x^{l-2}-+\cdots\} \qquad \ldots \quad \text{II}$$

So that we have $m+n=l$

$$c_1=a_1+b_1$$
$$c_2+1=a_2+1+b_2+1+a_1b_1$$
$$c_3=a_3+b_3+a_1\,(b_2+1)+b_1(a_2+1)$$
$$c_4+1=a_4+1+b_4+1+a_1b_3+a_3b_1+(a_2+1)(b_2+1)$$

etc.

Since the digit in the first place must be greater than zero,

$$10>a_1,\ b_1,\ c_1>0.$$

The digits in the other places are such that

$$10>a_{2r},\ b_{2r},\ c_{2r}\geqq 0\ ;\ 10\geqq a_{2r+1},\ b_{2r+1},\ c_{2r+1}>0$$

we can now construct any number of examples with the required property.

Let us take 71 as the multiplicand, and let $(10-b_1)\,10^2+b_2 10+(10-b_3)$ be its multiplier. The product is

$$(10^2-3\cdot 10+1)\;\{10^3-b_1\,10^2+(b_2+1)\,10-b_3\}.$$

Putting $x=10$ we get as in II

$$(x^2-3x+1)\;\{x^3-b_1x^2+(b^2+1)\,x-b_3\}$$
$$\equiv x^5-c_1x^4+(c_2+1)\,x^3-c_3x^2+(c_4+1)x-c_5.$$

where $10-c_1$, c_2, $10-c_3$, c_4 $10-c_5$ are the digits of the product from the left.

Our limitations then are

$$10>c_1=\quad 3+b_1\geqq 1$$
$$10>c_2=-1+b_2+1+3b_1+1\geqq 0$$
$$10\geqq c_3=\quad b_3+3b_2+3+b_1\geqq 1.$$
$$10>c_4=-1+3b_3+b_2+1\geqq 0$$
$$10\geqq c_5=\quad b_3\geqq 1.$$

Subject to these inequalities we may take

(1) $b_1=2,\ b_2=1,\ b_3=2$;

whence the product becomes $71 \times 818 = 58078$;

(2) $b_1=2,\ b_2=1,\ b_3=1$;

whence the product becomes $71 \times 819 = 58149$;

(3) $b_1=2,\ b_2=0,\ b_3=1$

whence we have $71 \times 809 = 57439$.

Other pairs of numbers can be obtained in the same way

Eg. $82 \times 708 = 58056$; $918 \times 709 = 650862$.

Remarks by the Proposer.

The Solution given by J. Mahrenholz in Vol. V, No. 4, p. 188 is only an algebraic rendering of the question, while in the other examples given by him the figures to be inserted are not 9 and 0 as required in the question.

Though I have not succeeded in getting a general formula, I have by continuous thinking over the matter for over a dozen years obtained empirically nearly a hundred such pairs with the property stated in the question of which I give a few below :—

$$81 \times 729 = 59049$$
$$71 \times 819 = 58149$$
$$81 \times 819 = 66339$$
$$92 \times 729 = 67068$$

In the two pairs given below, we have to regard groups of two digits as units between which groups of 9's and 0's are to be inserted. Thus

$$94\ 03\ 92 \times 93\ 02 = 87\ 47\ 52\ 63\ 84$$
$$994003992 \times 993002 = 987047952063984$$

We may also insert r nines and r zeros in the gaps instead of single nines and zeros.

Again $97\ 01\ 93 \times 98\ 01\ 92 = 95\ 09\ 75\ 41\ 70\ 56$ is another product with the same property.

Question 1758

(D. R. KAPREKAR)—Prove without actual multiplication that the last nine digits of huge number N given below are 999977778.

$$N = (546194538)\{1 + (24580) + (24580)^2 + \cdots + (24580)^{10}\}.$$

Solution by R. Sharangi.

$$N = 546194538 \times \{1 + 24580 + (24580)^2 + \cdot \cdot \cdot \cdot + (24580)^{10}\}$$

$$= 546194538 \times \left\{ \frac{(24580)^{11} - 1}{(24580) - 1} \right\}$$

$$= \frac{546194538}{24579} \times \left\{ (24580)^{11} - 1 \right\} = 22222 \left\{ (24580)^{11} - 1 \right\}$$

= a number ending in 11 zeros − 22222.

Hence the last nine digits of N are
999, 977, 778.

Also solved by H. Z. Gilami, B. Venkataramana Kolla, V. Narasimhamurti and J. Mahrenholz.

Question 1759 (Corrected)

(K. RANGASWAMI) :—If $x > 1$ and r, s are positive integers, then

$$\frac{x^r - x^{-r}}{r} \gtrless \frac{x^s - x^{-s}}{s}$$

according as $r \gtrless s$.

Solution by B. R. Venkataraman.

Let r be a positive integer. Then if $x > 0$ and $0 \leqslant k \leqslant 2r - 1$, we have

$$(x^{2r-k} - 1)(x^{k+1} - 1) > 0.$$

i.e., $x^{2r+1} + 1 > x^{2r-k} + x^{k+1}$

Giving the values 0, 1, 2, $\cdots\cdots$ $r-1$, for k in succession and adding up we get

$$r(x^{2r+1} + 1) > x + x^2 + \cdot \cdot \cdot \cdot + x^{2r}.$$

Hence, if $x > 1$, so that $x - 1$ is positive we have

$$r(x-1)(x^{2r+1} + 1) > x^{2r+1} - x.$$

i.e. $$(r+1)(x^{2r+1} - x) < r(x^{2r+2} - 1)$$

i.e. $$\frac{x^{2r+1} - x}{r} < \frac{x^{2r+2} - 1}{r+1}.$$

Multiplying both sides of the inequality by $x^{-(r+1)}$ we get,

$$\frac{x^r - x^{-r}}{r} < \frac{x^{r+1} - x^{-(r+1)}}{r+1}.$$

It follows immediately from the above that, when $x > 1$

$$\frac{x^r - x^{-r}}{r} \gtrless \frac{x^s - x^{-s}}{s} \text{ according as } r \gtrless s.$$

If we substitute $\frac{1}{x}$ for x in the above result the inequalities are reversed.

Thus, if $x<1$, $\frac{x^r - x^{-r}}{r} \lessgtr \frac{x^s - x^{-s}}{s}$ according as $r \gtrless s$.

Also solved by C. Radhakrishna Rao and R. Sharangi.

Question 1762

(R. RAMANUJAM)—If p, q, r represent the lengths of the altitudes of a triangle, find an expression for the area in terms of p, q, and r.

Solution by Johannes Mahrenholz.

Let a, b, c be the sides of the triangle and $\triangle$ its area.

Then $ap = bq = cr = 2\triangle$

Now, $16\triangle^2 = (a+b+c)(-a+b+c)(a-b+c)(a+b-c)$

$$= 16\triangle^4\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)\left(-\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}+\frac{1}{r}\right)\left(\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right).$$

$$\therefore \quad \triangle = p^2q^2r^2 \div \{(pq+qr+rp)(-pq+qr+rp)(pq-qr+rp)(pq+qr-rp)\}^{\frac{1}{2}}$$

Also solved by B. R. Venkataraman, C. Radhakrishna Rao, R. Sharangi, K. S. Ranganathan, S. N. Kawalgikar and analytically by B. G. Pandse.

REVIEWS

J. Tinbergen: *An Econometric Approach to Business Cycle Problems*: Hermann & Co., Paris, *1937, 73 pp. 18 francs.*

The usefulness of the mathematical method in economic research has not yet been fully realised in India, but in the West almost all the work in the field is done with this powerful tool. The book under review is an attempt at a mathematical treatment of the business cycle problems of the Netherlands.

The first of the three chapters of the book introduces the variables involved in the analysis. There are thirty-two of these consisting of prices, quantities, and values. They are connected among themselves by twenty-two regression equations which have been fitted in by the partial time regression method. These equations are valid only for the short run for, in their construction, as the author admits, long run influences such as the effect of new investments on productive capacity have been ignored. Again, it is assumed that workers spend the whole of their income whereas non-workers spend only a part of it. The most important of the conclusions reached by the author seems to be that the interest rate has not exerted much influence on the volume of investment. Investments and savings are found to be independent of each other for the short period but the author agrees with Mr. Keynes with regard to the long run relationship between the two.

The second chapter draws attention to the various problems in the history and the theory of business cycles that could be studied with the help of the variables and equations of the first chapter, and gives a detailed discussion of the relative importance of national and international influences in the theory. The author's conclusion is that the chief cyclical movements noticed are the imported ones.

The last chapter is concerned with business cycle policy and it is here that the author is at his best. He examines the various possible ways in which a Government might combat a depression. If the international economic position undergoes no change, the alternatives suggested are Investments on public works, Rationalisation, Lowering of monopoly prices, Wage reductions or Devaluation and the conclusion reached is that devaluation is the most favourable policy at least for the short run. If on the other hand international changes have to be reckoned with, the choice lies between stable exchange rates and stable prices. The latter will involve manipulation of public investment so as to make the total volume non-cyclical. The former leads to manipulation of exchange rates so as to equalise the price index for imports with its own trend value. The stabilising effect of the manipulation of exchange is found to be larger than that of the other. Coming to the question of the best policy, that is, the stabilisation of the level of employment or more generally the attainment of "the maximum of collective ophelimity" the author suggests changes in the existing coefficients wherever possible and the introduction of new coefficients like cost of living clauses in wage contracts. However he leaves the question open and concludes that his study only suggests a programme for further work and that what he has offered here must be regarded only as provisional findings.

The attention of the reader may now be drawn to one or two errors of omission and commission found in the book. The regression coefficients given have not been tested for significance and some of them are too small to permit any valid conclusions to be drawn from them. Secondly, many of the equations appear to have been over-simplified at the cost of accuracy, for instance, on pages 22 and 30. Thirdly in Chapter II on page 58, the author's conclusion that the movements of income consist of two cyclical components is not correct. The component dependent wholly on internal economy is given by $c_1x_1' + c_2x_2'$ where x_1 and x_2 are the roots of the equation $x^2 - \cdot 15x - \cdot 26 = 0$. It is clear that both of them are real but since they are proper fractions, this component becomes insignificant with the passage of time and the other one predominates. The fact is that the cyclical movement is entirely imported from abroad and this may entail changes in the arguments of the first part of Chapter III in the book.

Notwithstanding these defects, the reviewer has no doubt that this admirable treatise will be of great benefit to workers in the field of Trade Cycle theory.

Annamalainagar. S. SUBRAMANIAN.

M. V. JAMBUNATHAN, M.A., *Bala Thrikonamiti*, M. V. SESHADRI & Co., Bangalore, X + 100 pp. Price Annas Twelve.

This excellent booklet on Elementary Trigonometry by Rashtra Bhasha Visarada M.V. Jambunathan, Lecturer in Mathematics in the Intermediate College, Bangalore, seeks to give an introductory course in the subject up to the solution of right-angled triangles. The exposition is clear and illustrated with diagrams and worked examples, while there are a fair number of graded problems for the learner. The book should prove exceedingly useful to students preparing for the Mysore S.S.L.C. for whom it is primarily intended.

The preparation of technical books in the several Indian vernaculars is the first step towards the percolation of present-day scientific knowledge and experience to the masses and a powerful impetus to this movement is the progressive vernacularisation of the High School and the pre-Graduate courses which are being attempted in the various parts of the country today. While this makes for a more intimate acquaintance with the subject-matter, one should not altogether lose sight of the transition to a different language which is inevitable for one pursuing more advanced courses. A careful choice of notation and terminology with an eye on the whole of the country can help in softening this transition, and it is gratifying to note that the author has recognised this need. Occasionally, however, one meets a phrase which suggests something quite different, e.g. "Samakoṇa Trikona" which suggests an equilateral instead of a right-angled triangle.

For an elementary course, such as is intended by the author, the introduction of the general angle and its trigonometric ratios may well have been omitted. Possibly its inclusion is the result of the author reprinting the first few chapters of a bigger book under contemplation.

A table of natural signs, cosines and tangents at the end, a glossary of technical terms at the beginning enhance the value of the book.

A. NARASINGA RAO.

BOOKS RECEIVED FOR REVIEW

George Abraham MILLER: *Collected Works, University of Illinois, Urbana, Illinois.*

MILNE-THOMSON: *Theoretical Hydrodynamics*, Macmillan & Co., London.

Actualites Scientifiques et Industrielles, Hermann et Cie, Paris.

No. 552. Georges HOSTELET: *Les fondements experimentauxe de l' Analyse mathematique des faits statisques.*

No. 565. Moshen HACHTROUDI: *Les Espaces d' elements a connexion projective Normale.*

No. 570. George VALIRON: *Sur les Valeurs exceptionnelles des fonctions meromorphes et de leurs derivees.*

No. 571. Florin VASILESCO: *La notion de capacite.*

No. 578. C. de La Vallee POUSSIN: *Les nouvelles methodes de la Theorie du Potentiel et le Probleme generalise de Dirichlet.*

No. 585. Georges HOSTELET: *Le concours de l' Analyse mathematique à l'Analyse experimentale des faits statistiques.*

No. 589. Albert LAUTMAN: *Essai sur l' unite des sciences mathematiques dans leur developpement actuel.*

No. 590. Albert LAUTMAN: *Les schemas de structure.*

No. 591. Albert LAUTMAN: *Les schemas de genèse.*

ANNOUNCEMENTS AND NEWS

The following gentlemen have been elected members of the Society :—

J. L. Sharma, M.A., Lecturer, Christ Church College, Cawnpore.

Jageshwar Dayal Vaish, M.A., B.SC., Teacher, Sadul High School, Bikaner.

D. N. Moghe, Research Scholar, Fergusson College, Poona.

The Managing Committee of the Indian Mathematical Society has been reconstituted as follows for 1939-40.

President : *Dr. R. P. Paranjpye, M.A., D.Sc.*
Treasurer : *Prof. L. N. Subramaniam, M.A.*
Librarian : *Prof. R. P. Shintre, M.A.*

Other Members

1. *Dr. R. Vaidyanathaswamy, M.A., D. Sc., F.R.S.E.*
2. *The Hon'ble Dr. Sir Shah Mohamad Sulaiman, Kt., M.A., LL.D., D. Sc.*
3. *Dr. A. N. Singh, M. Sc., D. Sc.*
4. *Prof. Arunachala Sastri, M.A.*
5. *Prof. A. Narasinga Rao, M.A., L.T.*
6. *Dr. F. W. Levi.*
7. *Dr. P. L. Srivastava, M.A., D. Phil.*
8. *Dr. S. Chowla, M.A., Ph. D.*
9. *Dr. Ram Behari, M.A., Ph. D.*

With the financial co-operation of the Punjab University, the Indian Mathematical Society is printing Dr. Hansraj Gupta's Table of Partitions giving the values of $p(n)$ up to $n=600$ and also the values of (n, m), the number of partitions of n of which the smallest element that occurs is m, for values of n up to 300. This table has enabled Dr. Chowla to show that Ramanujan's conjecture regarding the divisibility of $p(n)$ by q^α where $q=5, 7$ or 11 and $24n-1$ is divisible by q^α, is not true when $q=7, \alpha=3$ (Vide: J. L. M. S. Vol. 9, p. 247).

The University of Illinois Press recently published Volume II of the Collected Works of Professor G. A. Miller. This volume includes an article of 36 pages on primary facts of the history of Mathematics and the correction of errors in text-books on this history, but the greater part of the volume is devoted to the publications of its author during the years from 1900 to 1908. These are practically confined to the theory of groups of finite order. This is the first time that an American university published the collected works of one of its faculty relating to pure mathematics.

We regret very much to announce the death on 12-11-1938 of Pandit Hemraj, M.A., Principal of the Dayal Singh College, Lahore. Pandit Hemraj has been a member of the Indian Mathematical Society since 1915 and was an active member of its managing committee for over 13 years.

QUESTIONS FOR SOLUTION

1765. (D. R. KAPREKAR). If A, B, C, D are four integers each containing the same numbers of digits such that $A+C=(9)^k$ and $B+D=(9)^k$, then

$$(A\times B)+(C\times D)+(A\times D)+(B\times C)+A+B+C+D=(9)^{2k}.$$

[The symbol $(9)^l$ is the integer formed by the digit 9 repeated l times.]

1766. (S. S. PILLAI). There are r bags each containing a fixed number of balls. By taking at most t bags at a time, we want to form groups containing any number of balls from 1 to n. Let $f(t, n)$ denote the least value of r for which this is possible.

Show that $f(t, n) = [(n-1)/(2^t - 1)] + t$.

1767. (W. F. KIBBLE): Evaluate

$$\begin{vmatrix} \frac{1}{k} & \frac{1}{k+a_1} & \frac{1}{k+a_1+a_2} & \cdots & \frac{1}{k+\sum_1^n a_r} \\ \frac{1}{k+b_1} & \frac{1}{k+a_1+b_1} & \frac{1}{k+a_1+a_2+b_1} & \cdots & \frac{1}{k+b_1+\sum a_r} \\ \frac{1}{k+b_1+b_2} & \frac{1}{k+a_1+b_1+b_2} & \frac{1}{k+a_1+a_2+b_1+b_2} & \cdots & \frac{1}{k+b_1+b_2+\sum a_r} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{k+\sum_1^n b_r} & \frac{1}{k+a_1+\sum b_r} & \cdots & \cdots & \frac{1}{k+\sum a_r+\sum b_r} \end{vmatrix}$$

and use the result to find

$$\begin{vmatrix} \frac{1}{k} & \frac{1}{k+1} & \frac{1}{k+2} & \cdots & \frac{1}{k+n} \\ \frac{1}{k+1} & \frac{1}{k+2} & \frac{1}{k+3} & \cdots & \frac{1}{k+n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{k+n} & \frac{1}{k+n+1} & \cdots & \cdots & \frac{1}{k+2n} \end{vmatrix}$$

and each of its minors.

1768. (R. RAMANUJAM). Construct geometrically a triangle given the lengths of the internal bisectors of the three angles.

1769. (A. NARASINGA RAO). ABC, A′B′C′ are two variable triangles all of whose sides tend to zero. It is given that the limits $\frac{AB}{A'B'}$, $\frac{BC}{B'C'}$, $\frac{CA}{C'A'}$, exist and are all equal to l. Show by examples, that the conclusion (suggested by similar triangles) that the limit of (area ABC/area A′B′C′) exists and is equal to l^2 is not valid. Obtain the further conditions to be fulfilled in order that the result may be true.

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